

THE TRANSVERSALITY CONDITIONS IN INFINITE HORIZON PROBLEMS AND THE STABILITY OF ADJOINT VARIABLE

Dmitry Khlopin

Received: 10.08.2011 / Accepted: date

Abstract This paper investigates the necessary conditions of optimality for uniformly overtaking optimal control on infinite horizon with free right endpoint. Clarke's form of the Pontryagin Maximum Principle is proved without the assumption on boundedness of total variation of adjoint variable. The transversality condition for adjoint variable is shown to become necessary if the adjoint variable is partially Lyapunov stable. The modifications of this condition are proposed for the case of unbounded adjoint variable. The Cauchy-type formula for the adjoint variable proposed by S. M. Aseev and A. V. Kryazhinskii in [1],[2] is shown to complement relations of the Pontryagin Maximum Principle up to the complete set of necessary conditions of optimality if the improper integral in the formula converges conditionally and continuously depends on the original position. The results are extended to an unbounded objective functional (described by a non-convergent improper integral), unbounded constraint on the control, and uniformly sporadically catching up optimal control.

Keywords Optimal control · infinite horizon problem · transversality condition for infinity · necessary conditions · Lyapunov stability · uniformly overtaking optimal

Mathematics Subject Classification (2000) 49K15 · 49J45 · 37N40 · 34D05

Introduction

The Pontryagin Maximum Principle for infinite horizon problems has already been formulated in the monograph [26], but without the transversality condition the obtained relations were incomplete and in general, selected a much too broad family of potentially extremal trajectories. A significant number [20, 4, 8, 25, 31,

Our grant could be here

D. Khlopin
Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, Yekaterinburg, Russia
E-mail: khlopin@imm.uran.ru

10,22] of such conditions has been proposed; however, as it was noted in, for example, [20,25,30],[4, Sect. 6],[28, Example 10.2], these conditions may be either inconsistent with the relations of the Pontryagin Maximum Principle, or follow from them. Hence the need to investigate the applicability of a transversality condition (see [4,8,22,25,31,36,28,29,27]) and the need to separately check if it is necessary for a specific optimization problem. The first aim of this paper is to offer a common approach to selecting a necessary transversality condition on the adjoint variable for this problem (Subsect. 4.3–4.4). However, the necessity of a condition does not imply its nontriviality on solutions of the relations of the Maximum Principle. Hence the need to find a condition that would select a single solution of the relations of the Maximum Principle for any uniformly overtaking optimal control. In the papers [1,2,3,4,5], Aseev and Kryazhinskii develop and investigate the Cauchy-type formula for the adjoint variable that possesses such a property. The second aim is to maximize the applicability of the approach [4] (Sect. 5).

First of all, we construct the bicomact extension (see [34]) for the space of admissible controls in the form of the inverse limit of the sequence of corresponding finite horizon extensions. It is shown that there exists a uniformly overtaking optimal generalized control for the case of a conditionally convergent objective functional that converges uniformly with respect to all trajectories; this generalizes some results [9],[12],[16]. Without this assumption, for uniformly overtaking optimal control for problems with free right endpoint, the necessity of the Pontryagin Maximum Principle in Clarke's form for the more general conditions than in [4,5,25],[36, Theorem 2.1] is shown; the obtained result is not a part of the results [7,20,27].

In Subsect. 3.3 for a free right endpoint problem, the convergence of transversality conditions on the adjoint variable is provided by the integral partial stability of the adjoint variable as a component of the Maximum Principle system. Thus we obtain the assumptions that guarantee the necessity of such condition, which are more general than the assumptions in [28],[36, Corollary 2.1],[7, Theorem 1] (It seems that the first implementation of the approach that employs the notion of stability to obtain transversality conditions was in [31]). If the the objective function and the right-hand side of the equation of dynamics are smooth (in the phase variable), then, instead of integral partial stability we can check the simpler condition of partial Lyapunov stability for the variable ψ as a component of solutions of the system of the Maximum Principle. For example, we can check if all Lyapunov exponents are negative for this variable (see Subsect. 4.3).

We propose the new transversality condition: the product of the adjoint variable and a matrix function of time must be vanishingly small at infinity. This condition becomes necessary if the product is stable. The stability can be provided by the correct choice of the matrix function; the choice may also reflect a priori information on stability and asymptotic estimates, which may allow to pinpoint the single extremal (see Subsect. 4.4).

If the above matrix function is the fundamental matrix of linearized system along the optimal trajectory, then, the corresponding transversality condition automatically yields the formula that was proposed for affine systems in [1],[2], general case of which was examined in [4, Theorems 11.1, 12.1], [5, Theorem 2],[7, Theorem 1]. As it was shown in [4, Sect. 16], the results of [8,36] are the corollaries of [4, Theorem 12.1],[5, Theorem 2].

Such choice of matrix function allows us to reduce the question of necessity of the corresponding condition to not just the question of the stability of the product, but even to the issue of checking if the improper integral from the formula [4, (12.8)], [5, Theorem 2], [7, Theorem 1] converges conditionally and continuously depends on the initial position of the original problem. This yields the Cauchy-type formula for the adjoint variable and the “normal” Pontryagin Maximum Principle under the assumptions weaker than in [4, Theorem 12.1]. This result also generalizes [27, Theorem 3.2], [36, Theorem 2.1] and [28, Theorems 3.1 and 8.1] (as far as the necessary conditions for problems with free right endpoint are concerned).

For the case of monotonous system, we also demonstrated certain estimates for the adjoint variable. In particular, we obtained the nonnegativity of adjoint variable under the weaker assumptions than in [6, Theorem 1], [35, Theorem 1], [4, Theorem 10.1].

In the last part of the paper, we extend the obtained results to the cases of σ -compact constraints on controls and to uniformly sporadically catching up optimal controls. Important breakthroughs for these problems were recently achieved in [7].

A part of the results of this paper has been shown and announced in paper [24].

1 Preliminaries

We consider the time interval $\mathbf{T} \triangleq \mathbf{R}_{\geq 0}$. The phase space of the original control system is the certain finite-dimensional metric space $\mathbf{X} \triangleq \mathbf{R}^m$. The unit ball of this space is denoted by \mathbf{D} . Let \mathbf{L} denote the linear space of all $m \times m$ matrices. For the sake of definiteness, let us equip \mathbf{L} with the operator norm. The symbol E (which may be equipped with some indices) denotes various auxiliary finite-dimensional Euclidean spaces, and the symbol $\mathcal{B}(E)$ denotes the σ -algebras of their Borel subsets.

For a subset A of a topological space, $cl A$ denotes the closure of this subset.

On the sets of all functions that are continuous on the whole \mathbf{T} , we consider the topology of uniform convergence on \mathbf{T} and the compact-open topology; for example, $C(\mathbf{T}, E)$ and $C_{loc}(\mathbf{T}, E)$. The first one is considered to be equipped with the norm $\|\cdot\|_C$ of the uniform convergence topology. Ω denotes the family of functions $\omega \in C(\mathbf{T}, \mathbf{T})$ such that $\lim_{t \rightarrow \infty} \omega(t) = 0$.

Here and below, for each summable function a of time, the integral $\int_{\mathbf{T}} a(t) dt$ is the limit $\int_{[0, T]} a(t) dt$ as $T \rightarrow \infty$. The integral over an infinite interval, for example, over $[T, \infty)$, is interpreted in the same sense.

Let us also consider a finite-dimensional Euclidean space \mathbf{U} and a set-valued map $U : \mathbf{T} \rightsquigarrow \mathbf{U}$. The set \mathfrak{U} of admissible controls is understood as the set of all Borel measurable selectors of the multi-valued map U . The topology on \mathfrak{U} is defined by virtue of the inclusion $\mathfrak{U} \subset \mathcal{L}_{loc}^1(\mathbf{T}, \mathbf{U})$.

A function $a : \mathbf{T} \times E' \times \mathbf{U} \rightarrow E''$ is said to

- 1) satisfy the Carathéodory conditions if a) the function $a(\cdot, y, u) : \mathbf{T} \rightarrow E''$ is measurable for all $(y, t, u) \in \mathbf{X} \times GrU$, b) the function $a(t, \cdot, \cdot) : E' \times U(t) \rightarrow E''$ is continuous for all $t \in \mathbf{T}$.

- 2) be locally Lipschitz continuous if for each compact $K \in (comp)(\mathbf{T} \times E)$ there exists a function $L_K^a \in \mathcal{L}_{loc}^1(\mathbf{T}, \mathbf{T})$ such that for all $(t, x'), (t, x'') \in K, u \in U(t)$, the inequality $\|a(t, x', u) - a(t, x'', u)\|_{E'} \leq L_K^a(t) \|x' - x''\|_{E'}$ holds.
- 3) be integrally bounded (on each compact subset of $\mathbf{T} \times E$) if for each compact $K \in (comp)(\mathbf{T} \times E)$ there exists a function $M_K^a \in \mathcal{L}_{loc}^1(\mathbf{T}, \mathbf{T})$ such that for all $(t, x) \in K, u \in U(t)$ we have $\|a(t, x, u)\|_{E''} \leq M_K^a(t)$.
- 4) satisfy the continuability condition on \mathbf{T} if it satisfies the sublinear growth condition, i.e., if the function f is Lipschitz continuous such that the function L_K^a is independent of K and is integrally bounded (on each compact subset); see [33, 1.4.6].

Here and below, we assume the following conditions hold:

Condition (u) : U is a compact-valued map such that it is integrally bounded (on each compact subset of \mathbf{T}) and $Gr U \in \mathcal{B}(\mathbf{T} \times \mathbf{U})$.

Condition (fg) : the mappings $f : \mathbf{T} \times \mathbf{X} \times \mathbf{U} \rightarrow \mathbf{X}, g : \mathbf{T} \times \mathbf{X} \times \mathbf{U} \rightarrow \mathbf{R}$ are locally Lipschitz continuous Carathéodory mappings that are integrally bounded (on each compact subset) and f satisfies the continuability condition.

Let us consider the control system

$$\dot{x} = f(t, x, u), \quad x(0) = 0, \quad t \in \mathbf{T}, \quad x \in \mathbf{X}, \quad u \in U(t). \quad (1a)$$

Now we can assign the solution (1a) to every $u \in \mathfrak{U}$. The solution is unique and it can be extended to the whole \mathbf{T} . Let us denote it by $\varphi[u]$. The mapping $\varphi : \mathfrak{U} \rightarrow C_{loc}(\mathbf{T}, \mathbf{X})$ is continuous.

In what follows, we examine the problem of maximizing the objective functional

$$\begin{aligned} \lim_{T \rightarrow \infty} J_T(u) &\rightarrow \max; \\ J_T(u) &\triangleq \int_0^T g(t, \varphi[u](t), u(t)) dt. \end{aligned} \quad (1b)$$

If there is no limit in (1b), the optimality may be defined in diverse ways (for details, see [13],[11],[32]), generally, we will use the following one:

Definition 1 A control $u^0 \in \mathfrak{U}$ is called uniformly overtaking optimal if for each $\varepsilon \in \mathbf{R}_{>0}$ there exists $T \in \mathbf{R}_{>0}$ such that $J_t(u^0) \geq J_t(u) - \varepsilon$ holds for all $u \in \mathfrak{U}$, $t \in [T, \infty)$.

Note that in paper [32], this definition is referred to as uniformly catching up optimal control. In [21, Theorem 3.1], it is shown that the uniformly overtaking optimality is equivalent to the condition

$$\lim_{t \rightarrow +\infty} \left(J_t(u^0) - \sup_{u \in \mathfrak{U}} J_t(u) \right) = 0$$

which, in terms of [21], says that u^0 is strongly agreeable.

Note that for each uniformly overtaking optimal control $u^0 \in \mathfrak{U}$ there exists a function $\omega^0 \in \Omega$ such that

$$J_t(u^0) \geq J_t(u) - \omega^0(T) \quad \forall u \in \mathfrak{U}, T \in \mathbf{T}, t \in [T, \infty). \quad (2)$$

2 On existence of uniformly overtaking optimal control

To complete the main objective of this section, we need the following assumption:

Condition (e) : there exists a function $\omega \in \Omega$ such that

$$\int_T^\tau g(t, \varphi[u](t), u(t)) dt \leq \omega(T) \quad \forall u \in \mathfrak{U}, T, \tau \in \mathbf{T}, T < \tau.$$

Note that to the best of author's knowledge, a one-sided condition like (e) was first proposed in paper [16, (II8)]. As it was actually proved in [16, Subsect 5.1], instead of (e), it is enough to assume, for example, the stronger condition

$$g(t, \varphi[u](t), u(t)) \leq l(t) \quad \forall u \in \mathfrak{U}, T \in \mathbf{T}$$

for some summable on \mathbf{T} mapping $l \in \mathcal{L}^1(\mathbf{T}, \mathbf{R})$.

2.1 The definition of the set $\tilde{\mathfrak{U}}$ of generalized controls

For each $u \in \mathbf{U}$, the symbol $\tilde{\delta}(u)$ denotes the probability measure concentrated at the point u . Let $\tilde{\mathfrak{U}}_n$ denote the family of all weakly measurable mappings μ from $[0, n]$ to the set of Radon probability measures over \mathbf{U} such that $\int_{U(t)} \eta(t)(du) = 1$ for a.a. $t \in [0, n]$. Let us equip this set with the topology of *-weak convergence. Then, the obtained topological space is a compact [34, IV.3.11], and the set $\tilde{\mathfrak{U}}_n \triangleq \{u|_{[0, n]} \mid u \in \mathfrak{U}\}$ is everywhere densely included in $\tilde{\mathfrak{U}}_n$ [34, IV.3.10] by the mapping $u \rightarrow \tilde{\delta} \circ u$.

Now, let us introduce the set of all maps η from \mathbf{T} into the set of Radon probability measures over \mathbf{U} such that $\eta|_{[0, n]} \in \tilde{\mathfrak{U}}_n$ for every $n \in \mathbf{N}$; and let us denote it by $\tilde{\mathfrak{U}}$. To each $n \in \mathbf{N}$ let projections $\tilde{\pi}_n : \tilde{\mathfrak{U}} \rightarrow \tilde{\mathfrak{U}}_n$ be given by $\tilde{\pi}_n(\eta) \triangleq \eta|_{[0, n]}$ for all $\eta \in \tilde{\mathfrak{U}}$. Let us equip $\tilde{\mathfrak{U}}$ with the weakest topology such that all projections are continuous. The set $\tilde{\mathfrak{U}}$ is called the set of generalized controls.

Let us assume that for the certain Euclidean space E a mapping $a : \mathbf{T} \times E \times \mathbf{U} \rightarrow (comp)(E)$ is given and the following condition is satisfied:

Condition (a) : the mapping $a : \mathbf{T} \times E \times \mathbf{U} \rightarrow (comp)(E)$ is a locally Lipschitz continuous integrally bounded Carathéodory mapping that satisfies the continuity condition.

Let us fix the set $\Xi \subset E$ of initial values and the system for $u \in \mathfrak{U}$:

$$\dot{y} = a(t, y(t), u(t)), \quad y(0) = \xi \in \Xi, \quad t \in \mathbf{T}, u \in \mathfrak{U}. \quad (3)$$

It can also be generalized for $\eta \in \tilde{\mathfrak{U}}$:

$$\dot{y} = \int_{U(t)} a(t, y(t), u) \eta(t)(du), \quad y(0) \in \Xi, \quad t \in \mathbf{T}, \eta \in \tilde{\mathfrak{U}}. \quad (4)$$

Each its local solution can be extended to the whole \mathbf{T} . For every $\eta \in \tilde{\mathfrak{U}}$, let us denote the family of all solutions $y \in C_{loc}(\mathbf{T}, E)$ of system (4) by $\mathfrak{A}[\eta]$.

2.2 The relaxed infinite-horizon optimal control problem.

Similarly, we can consider the solution $\tilde{\varphi}[\eta] \in C_{loc}(\mathbf{T}, \mathbf{X})$ of the Cauchy problem

$$\dot{x} = \int_{U(t)} f(\tau, x(\tau), u) \eta(t)(du), \quad x(0) = 0 \quad \forall \eta \in \tilde{\mathfrak{U}}, \quad (5a)$$

the function $T \mapsto \tilde{J}_T(\eta) \triangleq \int_{[0,T]} \int_{U(t)} g(t, \tilde{\varphi}[\eta](t), u) \eta(t)(du) dt$; and the problem of maximizing the functional

$$\lim_{T \rightarrow \infty} \tilde{J}_T(u) \rightarrow \max. \quad (5b)$$

Proposition 1 *Assume (u). Then,*

- 1) *the space $\tilde{\mathfrak{U}}$ is a compact, and $\tilde{\delta}(\mathfrak{U})$ is everywhere dense in it;*
- 2) *If (a) holds, then for a compact $\Xi \in (comp)(E)$ the map $\mathfrak{A} : \tilde{\mathfrak{U}} \rightarrow C_{loc}(\mathbf{T}, E)$ is continuous, and $\mathfrak{A}[\tilde{\delta} \circ \mathfrak{U}]$ is everywhere dense in $\mathfrak{A}[\tilde{\mathfrak{U}}] \in (comp)(C_{loc}(\mathbf{T}, E))$;*
- 3) *If (fg) hold, then $\tilde{\varphi}, \tilde{J} \in C_{loc}(\mathbf{T} \times \tilde{\mathfrak{U}}, \mathbf{R})$;*
- 4) *If (fg), (e) hold, then there is a uniformly overtaking optimal control $\tilde{u}^0 \in \tilde{\mathfrak{U}}$ for the relaxed problem (5a)–(5b) such that*

$$\begin{aligned} \lim_{T \rightarrow \infty} \sup_{u \in \mathfrak{U}} \int_0^T g(t, \varphi[u](t), u(t)) dt &= \lim_{T \rightarrow \infty} \max_{\eta \in \tilde{\mathfrak{U}}} \tilde{J}_T(\eta) = \max_{\eta \in \tilde{\mathfrak{U}}} \lim_{T \rightarrow \infty} \tilde{J}_T(\eta) = \\ &= \lim_{T \rightarrow \infty} \tilde{J}_T(\tilde{u}^0) = \int_{\mathbf{T}} \int_{U(t)} g(t, \tilde{\varphi}[\tilde{u}^0](t), u) \tilde{u}^0(du) dt, \end{aligned} \quad (6)$$

and all limits in (6) exist, although they can equal $-\infty$

Proof. For the sake of brevity, let us denote $\tilde{\Pi} \triangleq \prod_{n \in \mathbf{N}} \tilde{\mathfrak{U}}_n$, and let us equip it with Tikhonov topology. Let $\tilde{\Delta} : \tilde{\mathfrak{U}} \rightarrow \tilde{\Pi}$ be given by $\tilde{\Delta}(\eta) \triangleq (\tilde{\pi}_n(\eta))_{n \in \mathbf{N}}$ for all $\eta \in \tilde{\mathfrak{U}}$. It is a homeomorphism by continuity of the maps $\tilde{\pi}_n$ and $\tilde{\pi}_n \circ \tilde{\Delta}^{-1}$.

Let $n, k \in \mathbf{N}$, ($n > k$). Then, the space $\tilde{\mathfrak{U}}_n$ is included in $\tilde{\mathfrak{U}}_k$ by the mapping $\tilde{\pi}_k^n(\eta) \triangleq \eta|_{[0,k]}$ for all $\eta \in \tilde{\mathfrak{U}}_n$. By $\tilde{\pi}_k^n \circ \tilde{\pi}_i^k = \tilde{\pi}_i^n$ for all $n, k, i \in \mathbf{N}$, ($n > k > i$), we have the projective sequence of the topological spaces $\{\tilde{\mathfrak{U}}_n, \tilde{\pi}_k^n\}$; and we can define the inverse limit [18, III.1.5], [17, 2.5.1]. In our notation, we can write it in the form $\lim_{\leftarrow} \{\tilde{\mathfrak{U}}_n, \tilde{\pi}_k^n\} \triangleq \tilde{\Delta}(\tilde{\mathfrak{U}}) \subset \tilde{\Pi}$. As shown above, $\tilde{\Delta}$ is a homeomorphism; hence, $\tilde{\mathfrak{U}}$ is homeomorphous to $\tilde{\Delta}(\tilde{\mathfrak{U}})$. Now, by Kurosh Theorem [18, III.1.13], the inverse limit $\tilde{\Delta}(\tilde{\mathfrak{U}})$ of compacts $\tilde{\mathfrak{U}}_n$ is compact, and $\tilde{\mathfrak{U}}$ is a compact too. Similarly, from [17, 4.2.5] and [34, IV.3.11] it follows that $\tilde{\mathfrak{U}}$ is also metrizable.

Repeating the reasonings without \sim or referring to [17, 3.4.11] and [17, 2.5.6] yields $\mathfrak{U} \cong \lim_{\leftarrow} \{\mathfrak{U}_n, \pi_k^n\} \triangleq \Delta(\mathfrak{U}) \subset \Pi$.

For each $n \in \mathbf{N}$, let the mapping $e_n : \mathfrak{U}_n \rightarrow \tilde{\mathfrak{U}}_n$ be given by $e_n(u)(t) \triangleq (\tilde{\delta} \circ u)(t) = \tilde{\delta}_{u(t)}$ for all $t \in [0, n]$, $u \in \mathfrak{U}_n$. Since for all $n, k \in \mathbf{N}$, $n > k$ it holds that $e_k \circ \pi_k^n = e_n$, we have the projective system $\{e_n, \pi_k^n\}$. Passing to the inverse limit, we obtain the mapping $e_{\Delta} : \Delta(\mathfrak{U}) \rightarrow \tilde{\Delta}(\tilde{\mathfrak{U}})$; from $e_n \circ \pi_n = \tilde{\pi}_n \circ \tilde{\delta}$ we have $e_{\Delta} \circ \Delta = \tilde{\Delta} \circ \tilde{\delta}$, and

from $\tilde{\mathcal{U}}_n = cle_n(\mathcal{U}_n)$ ([34]) we have $\tilde{\Delta}(\tilde{\mathcal{U}}) = cle_{\Delta}(\Delta(\mathcal{U})) = cl(\tilde{\Delta} \circ \tilde{\delta})(\mathcal{U})$; now, by continuity of $\tilde{\Delta}^{-1}$, we obtain $\tilde{\mathcal{U}} = cl \tilde{\delta}(\mathcal{U})$.

The mapping $\tilde{\mathfrak{A}}[\eta]$ is continuous by virtue of, for example, [33, Theorem 3.5.6]; the set $\tilde{\mathfrak{A}}[\eta](\tilde{\mathcal{U}})$ is compact as a continuous image of a compact. In what follows, is sufficient to use $\tilde{\mathcal{U}} = cl \tilde{\delta}(\mathcal{U})$.

Replacing a and the compact Ξ with the mapping $\{(f, g)\}$ and the compact $\{(0_{\mathbf{X}}, 0_{\mathbf{R}})\}$, we obtain the continuous dependence on η for the maps $\tilde{\varphi}, \tilde{J}$. Now, by virtue of $cl J_t(\mathcal{U}) = cl \tilde{J}_t(\tilde{\delta} \circ \mathcal{U}) = \tilde{J}_t(\tilde{\mathcal{U}})$, the condition (e) holds for $\eta \in \tilde{\mathcal{U}}$ too, i.e., it holds that

$$\tilde{J}_t(\eta) \leq J_T(\eta) + \omega(T) \quad \forall \eta \in \tilde{\mathcal{U}}, T \in \mathbf{T}, t \in [T, \infty), \quad (7)$$

then,

$$\limsup_{t \rightarrow \infty} \tilde{J}_t(\eta) \leq J_T(\eta) + \omega(T) \quad \forall \eta \in \tilde{\mathcal{U}}, T \in \mathbf{T},$$

passing to the lower limit as $T \rightarrow \infty$, we obtain, for arbitrary $\eta \in \tilde{\mathcal{U}}$, the existence of the limit $\lim_{t \rightarrow \infty} \tilde{J}_t(\eta)$ (possibly infinite).

Then, for every $t \in \mathbf{T}$, there exists an $\eta_t \in \tilde{\mathcal{U}}$ such that

$$\mathfrak{R}_t \triangleq \max_{\eta \in \tilde{\mathcal{U}}} \tilde{J}_t(\eta) = \tilde{J}_t(\eta_t) \quad \forall t \in \mathbf{T}. \quad (8)$$

Since $(\eta_t)_{t \in \mathbf{T}}$ is in the compact, for the certain unbounded increasing sequence $(t_k)_{k \in \mathbf{N}} \in \mathbf{T}$ and the certain $\tilde{u}^0 \in \tilde{\mathcal{U}}$, it is $\eta_{t_k} \rightarrow \tilde{u}^0$. Let us also define

$$\overline{\mathfrak{R}} \triangleq \limsup_{t \rightarrow \infty} \mathfrak{R}_t, \underline{\mathfrak{R}} \triangleq \liminf_{t \rightarrow \infty} \mathfrak{R}_t, \mathfrak{R}^* \triangleq \sup_{\eta \in \tilde{\mathcal{U}}} \lim_{t \rightarrow \infty} \tilde{J}_t(\eta), \mathfrak{R}^0 \triangleq \lim_{t \rightarrow \infty} \tilde{J}_t(\tilde{u}^0). \quad (9)$$

Now,

$$\mathfrak{R}_{t_k} = \tilde{J}_{t_k}(\eta_{t_k}) \stackrel{(7)}{\leq} \tilde{J}_{t_i}(\eta_{t_k}) + \omega(t_i) \quad \forall i, k \in \mathbf{N} (i < k),$$

passing to the upper limit as $k \rightarrow \infty$ and then as $i \rightarrow \infty$, we obtain $\overline{\mathfrak{R}} \leq \tilde{J}_{t_i}(\tilde{u}^0) + \omega(t_i)$ and $\overline{\mathfrak{R}} \leq \mathfrak{R}^0$. Thus, for all $T \in \mathbf{T}$

$$\begin{aligned} \overline{\mathfrak{R}} \leq \mathfrak{R}^0 &\stackrel{(9)}{=} \lim_{t \rightarrow \infty} \tilde{J}_t(\tilde{u}^0) \leq \sup_{\eta \in \tilde{\mathcal{U}}} \lim_{t \rightarrow \infty} \tilde{J}_t(\eta) \stackrel{(9)}{=} \mathfrak{R}^* \\ &\stackrel{(7)}{\leq} \sup_{\eta \in \tilde{\mathcal{U}}} \tilde{J}_T(\eta) + \omega(T) \stackrel{(8)}{=} \mathfrak{R}_T + \omega(T). \end{aligned}$$

Passing to the lower limit as $T \rightarrow \infty$, we obtain $\overline{\mathfrak{R}} \leq \mathfrak{R}^0 \leq \mathfrak{R}^* \leq \underline{\mathfrak{R}}$, it remains to note that by virtue of $cl J_t(\mathcal{U}) = cl \tilde{J}_t(\tilde{\delta} \circ \mathcal{U}) = \tilde{J}_t(\tilde{\mathcal{U}})$, it holds that

$$\lim_{t \rightarrow \infty} \sup_{u \in \mathcal{U}} J_t(u) = \lim_{t \rightarrow \infty} \max_{\eta \in \tilde{\mathcal{U}}} \tilde{J}_t(\eta) = \lim_{t \rightarrow \infty} \mathfrak{R}_t = \mathfrak{R}^0 = \mathfrak{R}^*. \quad \square$$

Remark 1 As it was shown in 1), for each generalized control there exists the sequence of controls from \mathcal{U} that converges (in the topology $\tilde{\mathcal{U}}$) to it.

Remark 2 (turnpike property) Item 4) actually shows more. It shows that the uniformly overtaking optimal control \tilde{u}^0 can be obtained as a limitary point of the sets $\arg \max_{\eta \in \tilde{\mathcal{U}}} \tilde{J}_t(\eta) \in (comp)(\tilde{\mathcal{U}})$ as $t \rightarrow \infty$.

Remark 3 As it was shown in 4), the limit

$$\int_{\mathbf{T}} \int_{U(t)} g(t, \tilde{\varphi}[\eta](t), u) \eta(du) dt \triangleq \lim_{T \rightarrow \infty} \tilde{J}_T(\eta)$$

is defined (though it may be infinite) for all $\eta \in \tilde{\mathfrak{U}}$.

Note that not only did the paper [16] prove the theorem of existence of an optimal solution based on the condition (e) but it also discussed the proof of such theorems based on the inverse limit. To the best of author's knowledge, there is only one paper [23] besides the previous one in the control theory that explicitly employs the notion of inverse limit.

There are many existence theorems, for example, [9], [11], [13], [12]. The results obtained in Proposition 1 have much in common with paper [12] (in terms of [12], the obtained \tilde{u}^0 is strongly optimal). Note that if the initial set \mathfrak{U} does not contain a uniformly overtaking optimal control, we may pass to Gamkrelidze controls by increasing the dimension of the set \mathbf{U} in $m+1$ times. (For details of such bicomact extension, see [19], [12]). These controls also form a compact and the items 1)-3) of Proposition 1 hold from them; therefore, there always exists a uniformly overtaking optimal control among such finite-dimensional controls.

As a corollary, we assume the uniformly overtaking optimal control u^0 to exist among the elements of \mathfrak{U} , and denote the trajectory that corresponds to u^0 by x^0 . We also keep the denotation $\tilde{u}^0 \triangleq \tilde{\delta} \circ u^0$.

We are also interested in the degree of closeness of various generalized controls for large t . Let $w : \mathbf{T} \times \mathbf{U} \rightarrow \mathbf{T}$ be an integrally bounded Carathéodory map. For all $\tau \in \mathbf{T}$ and $\eta \in \tilde{\mathfrak{U}}$, let us introduce

$$\mathfrak{L}_w[\eta](\tau) \triangleq \int_0^\tau \int_{U(t)} w(t, u) \eta(du) dt.$$

Let us denote by $(Fin)(u^0)$ the family of $\eta \in \tilde{\mathfrak{U}}$ such that $\eta|_{[T, \infty)} = \tilde{u}^0|_{[T, \infty)}$ for the certain $T \in \mathbf{T}$. Let us assume that $\mathfrak{L}_w[\tilde{u}^0] \equiv 0$, and for every $\eta \in (Fin)(u^0)$ from $\mathfrak{L}_w[\eta](\tau) = 0$ for all $\tau \in \mathbf{T}$ it follows that η equals \tilde{u}^0 a.e. on $[0, \tau]$. The set of such w is denoted by $(Null)(u^0)$.

3 The necessary conditions of optimality

3.1 Relations of the Maximum Principle

Let the Hamilton–Pontryagin function $\mathcal{H} : \mathbf{X} \times GrU \times \mathbf{T} \times \mathbf{X} \rightarrow \mathbf{R}$ be given by

$$\mathcal{H}(x, t, u, \lambda, \psi) \triangleq \psi f(t, x, u) + \lambda g(t, x, u).$$

Let us introduce the relations

$$\dot{x}(t) = f(t, x(t), u(t)); \quad (10a)$$

$$\dot{\psi}(t) \in -\partial_x \mathcal{H}(x(t), t, u(t), \lambda, \psi(t)); \quad (10b)$$

$$\sup_{p \in U(t)} \mathcal{H}(x(t), t, p, \lambda, \psi(t)) = \mathcal{H}(x(t), t, u(t), \lambda, \psi(t)); \quad (10c)$$

$$x(0) = 0, \quad \|\psi(0)\|_{\mathbf{X}} + \lambda = 1. \quad (10d)$$

It is easily seen that for each $u \in \mathfrak{U}$, for each initial condition, system (10a)–(10b) has a local solution, and each solution of these relations can be extended to the whole \mathbf{T} .

Let us denote by \mathfrak{Y} the family of all solutions $(x, u, \lambda, \psi) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U} \times [0, 1] \times C_{loc}(\mathbf{T}, \mathbf{X})$ of system (10a)–(10b), (10d) on \mathbf{T} , and let us denote by \mathfrak{Z} the set of solutions from \mathfrak{Y} for which (10c) also holds a.e. on \mathbf{T} .

Let us introduce such conditions for generalized controls; namely, under initial condition (10d) let us consider

$$\dot{x}(t) = \int_{U(t)} f(t, x(t), u) \eta(t)(du); \quad (11a)$$

$$\dot{\psi}(t) \in - \int_{U(t)} \partial_x \mathcal{H}(x(t), t, u, \lambda, \psi(t)) \eta(t)(du); \quad (11b)$$

$$\sup_{p \in U(t)} \mathcal{H}(x(t), t, p, \lambda, \psi(t)) = \int_{U(t)} \mathcal{H}(x(t), t, u, \lambda, \psi(t)) \eta(t)(du). \quad (11c)$$

Similarly, for each $\eta \in \tilde{\mathfrak{U}}$ for each initial condition, system (11a)–(11b) has a local solution that can be extended to the whole \mathbf{T} .

Let us denote by $\tilde{\mathfrak{Y}}$ the family of all solutions $(x, \eta, \lambda, \psi) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbf{T}, \mathbf{X})$ of system (10d)–(11b). Let us also introduce $\tilde{\mathfrak{Z}}$, the family of $(x, \eta, \lambda, \psi) \in \tilde{\mathfrak{Y}}$ such that (11c) also holds a.e. on \mathbf{T} .

Let us note that for every $\eta \in \tilde{\mathfrak{U}}$, the family of all solutions $(x, \eta, \lambda, \psi) \in \tilde{\mathfrak{Y}}$ of system (10d)–(11a) on \mathbf{T} for given control η is compact by virtue of [33, Theorem 3.4.2]. Moreover, this compact-valued map is upper semicontinuous in η . Indeed, the right-hand side of (11a)–(11b) is convex and integrally bounded, upper semicontinuous in η , and it is measurable for each fixed x, ψ ; therefore, it has a measurable selector ([33, Lemm 2.3.11]); moreover, all local solutions of (11a)–(11b) can be extended to the whole \mathbf{T} . Since all the conditions of [33, Theorem 3.5.6] are satisfied, the mapping is upper semicontinuous. Therefore, $\tilde{\mathfrak{Y}}$ and $\tilde{\mathfrak{Z}}$ are compact, as the graphs of this mapping on the compact subdomain of its domain.

Note that by [14, Theorem 2.7.5] always holds the inclusion:

$$\partial_x \int_{U(t)} \mathcal{H}(x, t, u, \lambda, \psi) \eta(t)(du) \subset \int_{U(t)} \partial_x \mathcal{H}(x, t, u, \lambda, \psi) \eta(t)(du). \quad (12)$$

3.2 The necessity of the Maximum Principle

Theorem 1 *Assume conditions (u), (fg). For each uniformly overtaking optimal pair $(x^0, u^0) \in C(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ for problem (1a)–(1b), there exist $\lambda^0 \in [0, 1], \psi^0 \in C(\mathbf{T}, \mathbf{X})$ such that the relations of the Maximum Principle (10a)–(10d) hold; i.e., $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$.*

Proof.

Let us fix a certain unbounded monotonically increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$. Let us also consider an arbitrary sequence $(\gamma_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ that converges to zero with the property $\omega^0(\tau_n)/\gamma_n \rightarrow 0$, where the function ω^0 was taken from (2). For example, $\gamma_n \triangleq \sqrt{\omega^0(\tau_n)}$ will suffice.

Fix a $w \in (Null)(u^0)$. For each $n \in \mathbf{N}$ let us consider the problem

$$J_{\tau_n}(\eta) - \gamma_n \mathfrak{L}_w[\eta](\tau_n) = \int_0^{\tau_n} \int_{U(t)} g(t, \tilde{\varphi}[\eta](t), u) \eta(t)(du) dt - \gamma_n \mathfrak{L}_w[\eta](\tau_n) \rightarrow \max.$$

Here, the functional is bounded from above by the number $J_{\tau_n}(u^0) + \omega^0(\tau_n)$, therefore, it has the supremum. Every summand continuously depends on η , which covers the compact $\tilde{\mathfrak{U}}$; therefore, there is an optimal solution for this problem in $\tilde{\mathfrak{U}}$; let us denote one of them by (x^n, η^n) .

Let the function $\mathcal{H}_{\tau_n} : \mathbf{X} \times Gr U \times \mathbf{T} \times \mathbf{X} \rightarrow \mathbf{R}$ be given by

$$\mathcal{H}_{\tau_n}(x, t, u, \lambda, \psi) \triangleq \mathcal{H}(x, t, u, \lambda, \psi) - \gamma_n w(t, u).$$

Then, by the Clarke form [14, Theorem 5.2.1] of the Pontryagin Maximum Principle, there exists $(\lambda^n, \psi^n) \in \mathbf{T} \times C([0, n], \mathbf{X})$ such that relation (10d) and the transversality condition at the free endpoint $\psi^n(\tau_n) = 0$ hold, and

$$\begin{aligned} \sup_{p \in U(t)} \mathcal{H}_{\tau_n}(x^n(t), t, p, \lambda^n, \psi^n(t)) &= \int_{U(t)} \mathcal{H}_{\tau_n}(x^n(t), t, u, \lambda^n, \psi^n(t)) \eta^n(t)(du), \quad (13) \\ \dot{\psi}^n(t) &\in -\partial_x \int_{U(t)} \mathcal{H}_{\tau_n}(x^n(t), t, u, \lambda^n, \psi^n(t)) \eta^n(t)(du) \end{aligned}$$

also hold for a.a. $t \in [0, \tau_n]$. By (12), $(x^n, \eta^n, \lambda^n, \psi^n) \in \mathbf{T} \times C([0, n], \mathbf{X})$ satisfy the relations (10d)–(11b), (13) a. e. on the $[0, \tau_n]$.

Let us extend the $(x^n, \eta^n, \lambda^n, \psi^n)$ to $[\tau_n, \infty)$ by the generalized control $\tilde{u}^0|_{[\tau_n, \infty)}$. Then, $\eta^n \in (Fin)(u^0)$. Let us denote by \mathfrak{Z}^n the set of (x, u, λ, ψ) that satisfy relations (10d)–(11b) a. e. on \mathbf{T} , satisfy relation (13) a. e. on $[0, \tau_n]$, and possess the property $\tilde{u}^0|_{[\tau_n, \infty)} = \eta^n|_{[\tau_n, \infty)}$. Now we have $(x^n, \eta^n, \lambda^n, \psi^n) \in \mathfrak{Z}^n$ for every $n \in \mathbf{N}$.

Let us note that all \mathfrak{Z}^n are closed and, since these sets are contained in the compact $\tilde{\mathfrak{Y}}$, these sets are also compact. Hence, the sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbf{N}}$ has the limit point $(x^{00}, \eta^0, \lambda^0, \psi^0) \in \tilde{\mathfrak{Y}}$. Passing, if necessary, to a subsequence, we may assume that it is the limit of the sequence itself.

For a fixed x , the set of $u \in U(t)$ that realize the maximum in (13) has a measurable selector by virtue of [15, Theorem 3.7]. By [33, Lemm 2.3.11], it exists if we put an arbitrary continuous function x into \mathcal{H} . Besides, since relation (13) also depends on x, ψ and on the parameters γ and λ upper semicontinuously, and all the relations are integrally bounded on bounded sets; by virtue of [33, Theorem 3.5.6], on each finite interval for the funnels of solutions of (10a)–(10b) that satisfy (13), we have upper semicontinuity by γ, λ . In particular, for $\gamma_n \rightarrow 0, \lambda^n \rightarrow \lambda^0$, we obtain the fact that the upper limit of the compacts \mathfrak{Z}^n is included in $\tilde{\mathfrak{Z}}$. Hence, $(x^{00}, \eta^0, \lambda^0, \psi^0) \in \tilde{\mathfrak{Z}}$.

On the other side, by $w \in (Null)(u^0)$ and by optimality of η^n, u^0 for their problems, we obtain

$$\tilde{J}_{\tau_n}(\eta^n) - \gamma_n \mathfrak{L}_w[\eta^n](\tau_n) \geq J_{\tau_n}(u^0) \stackrel{(2)}{\geq} \tilde{J}_{\tau_n}(\eta^n) - \omega^0(\tau_n)$$

therefore, we have $\gamma_n \mathfrak{L}_w[\eta^n](\tau_n) \leq \omega^0(\tau_n)$. By virtue of $\tilde{u}^0|_{[\tau_n, \infty)} = \eta^n|_{[\tau_n, \infty)}$, we obtain

$$\mathfrak{L}_w[\eta^n](\tau) \leq \omega^0(\tau_n)/\gamma_n \quad \forall \tau \in \mathbf{T}. \quad (14)$$

For each $\tau \in \mathbf{T}$, passing to the limit as $n \rightarrow \infty$, we obtain that $\mathfrak{L}_w[\eta^0] \leq 0$; i.e., $\mathfrak{L}_w[\eta^0](\tau) = 0$ for all $\tau \in \mathbf{T}$. Since $w \in (Null)(u^0)$, we have $\eta^0 = \tilde{u}^0$ a.e. on \mathbf{T} , hence $x^{00} = x^0$ and $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$. Moreover, from (14), we have $\|\mathfrak{L}_w[\eta^n]\|_C \rightarrow 0$. \square

We have additionally proved that

Remark 4 Under conditions **(u)**, **(fg)**, for each optimal pair $(x^0, u^0) \in \mathfrak{X} \times \mathfrak{U}$ for problem (1b), for each weight $w \in (Null)(u^0)$, for each unbounded increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$, we have constructed the sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbf{N}} \in \tilde{\mathfrak{Y}}^{\mathbf{N}}$ that possesses the following properties:

- 1) This sequence (as a sequence from $C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times \mathbf{T} \times C_{loc}(\mathbf{T}, \mathbf{X})$) converges to the certain $(x^0, \tilde{u}^0, \lambda^0, \psi^0) \in \mathfrak{Z}$;
- 2) $\|\mathfrak{L}_w(\eta^n)\|_C \rightarrow 0$;
- 3) $J_{t_n}(\eta) - J_{t_n}(u^0) \rightarrow 0$, and $\psi^n(t_n) = 0$ for each $n \in \mathbf{N}$, where $(t_n)_{n \in \mathbf{N}}$ is a certain subsequence of $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$.

3.3 The simplest condition of transversality

However, the relations of the Maximum Principle are incomplete, since (10a)–(10d) do not contain a condition on the right endpoint. There are several variants of such additional conditions (for details, see [4, Sect. 6,12],[29]); in this paper we investigate the modifications of the condition

$$\lim_{t \rightarrow \infty} \psi(t) = 0. \quad (15a)$$

Let us formulate the propositions in terms of the stability of ψ such that a condition would be necessary.

Condition (ψ) : There exists a weight $w \in (Null)(u^0)$ such that for every solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$, the Lagrange multiplier ψ^0 is stable under \mathfrak{L}_w —small perturbations of system (10a)–(10b); i.e., for every $\varepsilon \in \mathbf{R}_{>0}$, there exist a number $\delta \in \mathbf{R}_{>0}$ and a neighborhood $\mathcal{Y} \subset C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbf{T}, \mathbf{X})$ of the solution $(x^0, \tilde{u}^0, \lambda^0, \psi^0)$ such that for every solution $(x, \eta, \lambda, \psi) \in \mathcal{Y} \cap \tilde{\mathfrak{Y}}$ from $\|\mathfrak{L}_w[\eta]\|_C < \delta$ it follows that $\|\psi - \psi^0\|_C < \varepsilon$.

Proposition 2 Assume conditions **(u)**, **(fg)** hold. For each uniformly overtaking optimal pair $(x^0, u^0) \in C(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ satisfying (ψ) , for each unbounded increasing sequence, $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ there exists $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ such that

$$\liminf_{n \rightarrow \infty} \|\psi^0(\tau_n)\|_{\mathbf{X}} = 0 \quad (15b)$$

holds.

Proof. Let us choose the certain $\varepsilon \in \mathbf{R}_{>0}$, and let us take $\mathcal{Y} \subset \tilde{\mathfrak{Y}}$ and $\delta \in \mathbf{R}_{>0}$ from condition (ψ) ; by Remark 4, there exists $N \in \mathbf{N}$ such that for $n \in \mathbf{N}, n > N$, it is $(x^n, \eta^n, \lambda^n, \psi^n) \in \mathcal{Y}$, $\|\mathfrak{L}_w[\eta^n]\|_C < \delta$; now, condition (ψ) also yields $\|\psi^n(\tau_n) - \psi^0(\tau_n)\|_{\mathbf{X}} < \varepsilon$; but $\psi^n(\tau_n) = 0$; whence $\|\psi^0(\tau_n)\|_{\mathbf{X}} < \varepsilon$ for all $n \in \mathbf{N}, n > N$. Since $\varepsilon \in \mathbf{R}_{>0}$ was arbitrary, we have shown (15b). \square

Note that by linearity of (10b), the stability of the variable ψ implies its boundedness. Therefore, the proved proposition is useless for unbounded adjoint variable ψ .

Note that, as it follows from [32, Example 5.1], for a uniformly overtaking optimal control, there can be no $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ that satisfies stronger condition (15a) instead of (15b). On the other side,

Remark 5 Assume the functions L_K^f, L_K^g are independent of a compact K , and the mapping $T \mapsto L_K^g(T)e^{\int_{[0,T]} L_K^f(t)dt}$ is summable on \mathbf{T} ([27, Hypotesis 3.1 (iv)]); therefore, the total variation of ψ is a fortiori bounded. Then, (ψ) holds and, moreover, (15b) implies (15a).

The even more strong conditions used for proving the Maximum Principle can be seen, for example, in [36, (A3)] (the Lipschitz constants were required to decrease exponentially with time). Naturally, the propositions proved there for the condition are also covered by proposition 2.

One of the most general conditions on (15a) was shown in [28]. For a control problem without phase restrictions, the transversality condition from [28, Theorem 6.1] follows from Proposition 2 and [28, Lemm 3.1], or from Remark 5 and condition [28, (C3)]. The Remark 4 automatically yields [28, Theorem 8.1].

4 The necessity and the stability

The objective at hand is to choose the weight $w^0 \in (Null)(u^0)$ such that condition (ψ) would follow from a variety of (nonasymptotic) Lyapunov stability of ψ .

4.1 On weight w^0

Assume conditions $(\mathbf{u}), (\mathbf{a})$ hold. In what follows, assume $\Xi \triangleq E$. Then, for every position $(\tau^*, y^*) \in \mathbf{T} \times E$ there exists the unique solution y^0 of the equation

$$\dot{y} = a(t, y(t), u^0(t)), \quad y(\tau^*) = y^*, \tau^* \in \mathbf{T} \quad (16)$$

that can be extended to the whole time interval \mathbf{T} . It (as an element of $\tilde{\mathfrak{A}}(\tilde{u}^0) \subset C_{loc}(\mathbf{T}, E)$) continuously depends on $(\tau^*, y^*) \in \mathbf{T} \times E$. Let us denote its initial position $y^0(0)$ by $\varkappa(\tau^*, y^*)$.

Proposition 3 *Assume $(\mathbf{u}), (\mathbf{a})$ hold. Let the compact-valued map $G : \mathbf{T} \rightsquigarrow E$ be bounded on each compact set, and let $Gr G$ be closed.*

Then, there exists $w^0 \in (Null)(u^0)$, such that for arbitrary $\eta \in \tilde{\mathfrak{A}}$, $T \in \mathbf{T}$ for every $y \in \tilde{\mathfrak{A}}[\eta]$ from $Gr y|_{[0,T]} \subset Gr G$ it follows that

$$\|\varkappa(\tau, y(\tau)) - y(0)\|_E \leq \mathfrak{L}_{w^0}[\eta](\tau) \quad \forall \tau \in [0, T].$$

Proof.

Fix an $n \in \mathbf{N}$. By continuability, for each $(\tau^*, y^*) \in Gr G|_{[0,n]}$, there exists the position $\varkappa(\tau^*, y^*)$; by virtue of the theorem of continuous dependence on initial conditions, this mapping is continuous; hence, the image

$$\bar{G}_n \triangleq \left\{ e \in Gr y|_{[0,n]} \mid \forall y \in \tilde{\mathfrak{A}}[\tilde{u}^0], (\tau^*, y(\tau^*)) \in Gr G|_{[0,n]} \right\}$$

is closed; by the continuability, this set is bounded and, therefore, compact. Therefore, on this set, the function $a(t, y, u^0(t))$ is Lipschitz continuous with respect to y for the certain Lipschitz constant $L_n \triangleq L_{\bar{G}_n}^a \in \mathcal{L}_{loc}^1(\mathbf{T}, \mathbf{T})$. For all $t \in [0, n]$, define $M_n(t) \triangleq \int_{[0,t]} L_n(\tau) d\tau$. Note that this function is absolutely continuous and monotonically nondecreasing.

Fix $n \in \mathbf{N}$; for all $t \in [n-1, n]$, $u \in \mathbf{U}$, let us consider a number

$$R(t, u) \triangleq \sup_{y \in \bar{G}_n} \|a(t, y, u) - a(t, y, u^0(t))\|_E.$$

Note that the norm inside is a mapping that is continuous with respect to y and u , and y assumes values from the compact set; now, for every $u \in \mathbf{U}$ by [15, Theorem 3.7] the supremum reaches the maximum for the certain function $y_{max}[u] \in \mathcal{L}^1([n, n-1], \bar{G}_n)$. Hence, $R(t, u)$ is measurable with respect to t for each $u \in \mathbf{U}$.

Fix a $t \in [n-1, n]$; for each sufficiently small neighborhood $\Upsilon \subset U(t)$, by continuity of $a(t, \cdot, \cdot)$ on compact $\bar{G}_n \times cl\Upsilon$, there exists a function $\omega^t \in \Omega$, for which

$$\left| \|a(t, y, u') - a(t, y, u^0(t))\| - \|a(t, y, u'') - a(t, y, u^0(t))\| \right| < \omega^t \left(\frac{1}{\|u' - u''\|} \right) \quad (17)$$

holds for every $y \in \bar{G}_n, u', u'' \in \Upsilon (u' \neq u'')$. Without loss of generality, assume $R(t, u') \leq R(t, u'')$. Now, by definition, $R(t, u') \geq \|a(t, y, u') - a(t, y, u^0(t))\|$, and, substituting $y \triangleq y_{max}(u'')(t)$ into (17), we obtain $0 \leq R(t, u'') - R(t, u') \leq \omega^t(1/\|u' - u''\|)$; i.e., R is continuous with respect to the variable u on each sufficiently small neighborhood $\Upsilon \subset U(t)$; therefore on $U(t)$ and $GrU|_{[n-1,n]}$ too. Thus, the function $R : GrU|_{[n-1,n]} \rightarrow \mathbf{T}$ is a Carathéodory function.

Let us note that by considering all $n \in \mathbf{N}$, we define the Carathéodory function R on the whole GrU . Moreover, by construction, $R(t, u^0(t)) \equiv 0$. Hence, it is correct to define $w^0 \in (Null)(u^0)$ by the rule

$$w^0(t, u) \triangleq \|u - u^0(t)\| + e^{M_n(t)} R(t, u) \quad \forall n \in \mathbf{N}, (t, u) \in GrU|_{[n-1,n]}.$$

Consider arbitrary $n \in \mathbf{N}$, $\tau^* \in [0, n]$, and $(\tau^*, y_1^*), (\tau^*, y_2^*) \in \bar{G}_n$. For the solutions $y_1, y_2 \in \tilde{\mathfrak{A}}[\tilde{u}^0]$ of equation (16), for the initial conditions $y_i(\tau^*) = y_i^*$, we have $Gr y_i|_{[0,n]} \subset \bar{G}_n$. Let us introduce functions

$$r(t) \triangleq y_1(t) - y_2(t), \quad W_+(t) \triangleq e^{M_n(t)} \|r(t)\|_E \quad \forall t \in [0, n].$$

By Lipschitz continuity of the right-hand side of (16) we obtain $\|\dot{r}(t)\|_E \geq -L_n(t)\|r(t)\|_E$, and

$$\frac{dW_+^2(t)}{dt} = 2L_n(t)W_+^2(t) + 2e^{2M_n(t)}r(t)\dot{r}(t) \geq 2L_n(t)W_+^2(t) - 2L_n(t)W_+^2(t) = 0.$$

Thus, the function W_+ is nondecreasing, and finally for all $(\tau, y_1^*), (\tau, y_2^*) \in \bar{G}_n$ we have

$$\|\varkappa(\tau, y_1^*) - \varkappa(\tau, y_2^*)\|_E = W_+(0) \leq W_+(\tau) = e^{M_n(\tau)} \|y_1^* - y_2^*\|_E. \quad (18)$$

Assume the $\eta \in \tilde{\mathfrak{U}}$, $y \in \tilde{\mathfrak{A}}[\eta]$, $T \in \mathbf{T}$ satisfy $Gr y|_{[0, T]} \subset Gr G$. Fix arbitrary $n \in \mathbf{N}$ and $\tau_1, \tau_2 \in [0, T] \cap [n-1, n]$, $\tau_1 < \tau_2$. There exists the solution $y^0 \in \tilde{\mathfrak{A}}[\tilde{u}^0]$ that satisfies the condition $y^0(\tau_1) = y(\tau_1)$; let us also define

$$r \triangleq y^0(t) - y(t), \quad W_-(t) \triangleq e^{-M_n(t)} \|r(t)\|_E \quad \forall t \in [\tau_1, \tau_2].$$

By construction of \bar{G}_n , we have $Gr y|_{[\tau_1, \tau_2]}, Gr y^0|_{[\tau_1, \tau_2]} \subset \bar{G}_n$. Now,

$$\begin{aligned} \frac{dW_-^2(t)}{dt} &= 2e^{-2M_n(t)} r(t) \dot{r}(t) - 2L_n(t) W_-^2(t) = \\ &= 2e^{-2M_n(t)} r(t) (\dot{y}^0(t) - a(t, y(t), u^0(t)) + a(t, y(t), u^0(t)) - \dot{y}(t)) - 2L_n(t) W_-^2(t) \leq \\ &= 2e^{-2M_n(t)} \|r(t)\|_E \int_{U(t)} R(t, u) \eta(t) (du) + 2L_n(t) W_-^2(t) - 2L_n(t) W_-^2(t) \leq \\ &= 2e^{-M_n(t)} W_-(t) \int_{U(t)} R(t, u) \eta(t) (du) \leq 2e^{-2M_n(t)} W_-(t) \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt}. \end{aligned}$$

Since function W_- is nonnegative, for a. a. $t \in \{t \in [\tau_1, \tau_2] \mid W_-(t) \neq 0\}$ we obtain

$$\frac{dW_-(t)}{dt} \leq e^{-2M_n(t)} \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt} \leq e^{-2M_n(\tau_1)} \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt}. \quad (19)$$

This inequality is trivial for $[\tau_1, \tau_2] \ni t < \sup\{t \in [\tau_1, \tau_2] \mid W_-(t) = 0\}$; whence,

$$\begin{aligned} \|\varkappa(\tau_2, y^0(\tau_2)) - \varkappa(\tau_2, y(\tau_2))\|_E &\stackrel{(18)}{\leq} e^{M_n(\tau_2)} \|y^0(\tau_2) - y(\tau_2)\|_E = \\ &= e^{2M_n(\tau_2)} W_-(\tau_2) \stackrel{(19)}{\leq} e^{2M_n(\tau_2) - 2M_n(\tau_1)} (\mathfrak{L}_{w^0}[\eta](\tau_2) - \mathfrak{L}_{w^0}[\eta](\tau_1)). \end{aligned}$$

But $\varkappa(\tau_2, y^0(\tau_2)) = y^0(0) = \varkappa(\tau_1, y^0(\tau_1)) = \varkappa(\tau_1, y(\tau_1))$, hence, we have

$$\|\varkappa(\tau_2, y(\tau_2)) - \varkappa(\tau_1, y(\tau_1))\|_E \leq e^{2M_n(\tau_2) - 2M_n(\tau_1)} (\mathfrak{L}_{w^0}[\eta](\tau_2) - \mathfrak{L}_{w^0}[\eta](\tau_1)). \quad (20)$$

Fix arbitrary $t \in [0, T]$. For each $\varepsilon \in \mathbf{R}_{>0}$ we can split interval $[0, t]$ into the intervals of the form $[\tau', \tau'']$ such that $M_n(\tau'') - M_n(\tau') = \int_{[\tau', \tau'']} L_n(t) dt < \varepsilon$ and $[\tau', \tau''] \subset [n-1, n]$ for the certain $n \in \mathbf{N}$. But, (20) holds for every interval, i.e.,

$$\|\varkappa(\tau'', y(\tau'')) - \varkappa(\tau', y(\tau'))\|_E \leq e^{2\varepsilon} (\mathfrak{L}_{w^0}[\eta](\tau'') - \mathfrak{L}_{w^0}[\eta](\tau')).$$

Summing for all intervals, by $\varkappa(0, y(0)) = y(0)$ and by the triangle inequality, we obtain $\|\varkappa(t, y(t)) - y(0)\|_E \leq e^{2\varepsilon} \mathfrak{L}_{w^0}[\eta](t)$ for every $t \in [0, T]$. Arbitrariness of $\varepsilon \in \mathbf{R}_{>0}$ completes the proof of the proposition. \square

4.2 The partial Lyapunov stability

Assume E can be represented in the form $E = E_p \times E_q$ for some finite-dimensional Euclidean subspaces E_p and E_q . Let us denote the projections of the map a to the subspaces E_p and E_q by b and c , respectively. Now, the system (3) can be written in the form

$$\dot{p} = b(t, p, q, u), \quad \dot{q} = c(t, p, q, u), \quad (p, q)(0) = \xi \in E, \quad u \in U(t); \quad (21)$$

Then, it is possible to say that for all $\eta \in \tilde{\mathfrak{U}}$, the set $\tilde{\mathfrak{A}}[\eta]$ contains pairs of functions $(p, q) \in C_{loc}(\mathbf{T}, E_p) \times C_{loc}(\mathbf{T}, E_q)$. For every $\xi \in E$, let us denote by $y_\xi^0 \triangleq (p_\xi^0, q_\xi^0) \in \tilde{\mathfrak{A}}[u^0]$ the unique solution of (16) for $\tau^* = 0, y^* = \xi$.

Definition 2 Consider a closed set $G_0 \subset E$ and $\xi \in G_0$. We say that the solution y_ξ^0 of equation (16) has Lyapunov stable component p_ξ^0 in domain G_0 if for each $\varepsilon \in \mathbf{R}_{>0}$ there exists $\delta(\varepsilon, y) \in \mathbf{R}_{>0}$ such that for each $\xi' \in G_0$ from $\|\xi' - \xi\|_E < \delta(\varepsilon, y)$ it follows that $\|p_{\xi'}^0(s) - p_\xi^0(s)\|_E < \varepsilon$ for all $s \in \mathbf{T}$.

Proposition 4 Assume (u), (a) holds. Suppose there is a closed set $G_0 \subset E$ and a compact $K_0 \in (comp)(G_0)$ such that for each $\xi \in K_0$ the solution y_ξ^0 of equation (16) has Lyapunov stable component p_ξ^0 in G_0 .

Then, for each $\varepsilon \in \mathbf{R}_{>0}$, there exists a number $\delta \in \mathbf{R}_{>0}$ such that for all $\eta \in \tilde{\mathfrak{U}}, y = (p, q) \in \tilde{\mathfrak{A}}[\eta]$ from $y(0) \in K_0, \|L_{w^0}[\eta]\|_C < \delta$, and $\varkappa(t, y(t)) \in G_0$ for all $t \in \mathbf{T}$, it follows that $\|p - p_{y(0)}^0\|_C < \varepsilon$.

Proof. Consider a compact $K_> \triangleq \{\xi \in G_0 \mid \exists \xi_0 \in K_0 \|\xi - \xi_0\|_E \leq 1\}$. To each $t \in \mathbf{T}$, let us assign the set $G(t) \triangleq \{y(t) \mid \eta \in \tilde{\mathfrak{U}}, y \in \tilde{\mathfrak{A}}[\eta], y(0) \in K_>\}$. The obtained map G is compact-valued and continuous; in particular, its graph is closed. Now we can use Proposition 3 for the multi-valued map G and fix the weight $w^0 \in (Null)(u^0)$ which exists by this Proposition.

Define

$$M(\xi', \xi'') \triangleq \sup_{t \in \mathbf{T}} \|p_{\xi'}^0(t) - p_{\xi''}^0(t)\|_{E_p} \in \mathbf{T} \cup \{+\infty\} \quad \forall \xi', \xi'' \in K_>.$$

For all $\xi \in K_0$, the stability of the component p_ξ^0 implies that the map M is finite and continuous at the point $(\xi, \xi) \in K_> \times K_>$.

Fix an $\varepsilon \in \mathbf{R}_{>0}$; choose for every $\xi \in K_0$ its $\delta(\varepsilon/2, y_\xi^0) \in \langle 0, 1/2 \rangle$; now, we have also chosen the $\delta(\varepsilon/2, y_\xi^0)$ -neighborhood of the point (ξ, ξ) (in $K_> \times K_>$). From the obtained cover of the diagonal Δ of the set $K_0 \times K_0$, let us select a finite subcover; it induces certain open neighborhood \mathcal{Y} of the diagonal Δ . Let $\delta(K_0)$ be the minimum distance from the diagonal Δ to the boundary of the neighborhood \mathcal{Y} . Now, for all $\xi' \in K_>, \xi \in K_0$ from $\|\xi' - \xi\|_E < \delta(K_0)$ it follows that $(\xi', \xi) \in \mathcal{Y}$; i.e., for some $\xi'' \in K_0$ we have $M(\xi, \xi''), M(\xi'', \xi') < \varepsilon/2$, whence $M(\xi, \xi') < \varepsilon$. Thus,

$$(\|\xi' - \xi\|_E < \delta(K_0)) \Rightarrow (\|p_{\xi'}^0 - p_\xi^0\|_C < \varepsilon) \quad \forall \xi \in K_0, \xi' \in K_>. \quad (22)$$

Suppose the $\eta \in \tilde{\mathfrak{U}}, y = (p, q) \in \tilde{\mathfrak{A}}[\eta]$ satisfy $L_{w^0}[\eta](t) < \delta, \xi_1(t) \triangleq \varkappa(t, y(t)) \in G_0$ for all $t \in \mathbf{T}$. For $K_0 \subset K_> = G(0)$, the definition

$$T_0 \triangleq \sup\{T \in \mathbf{T} \mid Gr \xi_1|_{[0, T]} \subset K_>\} \quad \forall t \in [0, T) \in \mathbf{T} \cup \{+\infty\}$$

is correct, although T_0 can be infinite. Hence, we have $Gr y|_{[0,t]} \subset Gr G$ for all $t \in [0, T]$. Now, from Proposition 3, we obtain

$$\|\xi_1(t) - y(0)\|_E = \|\varkappa(t, y(t)) - y(0)\|_E \leq \mathfrak{L}_{w^0}[\eta](t) < \delta(K_0) \quad \forall t \in [0, T_0]. \quad (23)$$

For every $t \in [0, T_0]$, let us substitute $\xi = y(0), \xi' \triangleq \xi_1(t) \in K_>$ in (22); from the equality $p_{\xi_1(t)}^0(t) = p(t)$ we obtain $\|p(t) - p_{y(0)}^0(t)\|_E < \varepsilon$ for all $[0, T_0]$. To conclude the proof, it remains to prove that $T_0 = \infty$.

Suppose $T_0 \in \mathbf{T}$; by construction of T_0 , for each $\tau \in \langle T_0, \infty \rangle$, we have $Gr \xi_1|_{\langle T_0, \tau \rangle} \not\subset Gr K_>$; but $Gr \xi_1 \subset G_0$. Then, $\rho(\xi_1(T_0), G_0 \setminus K_>) = 0$, and, in particular, by construction of $K_>$, we have $\|\xi_1(T_0) - y(0)\|_E \geq 1$. However, passing to the limit in (23) yields $\|\xi_1(T_0) - y(0)\|_E \leq \delta(K_0) \leq 1/2$. The acquired contradiction proves that $T_0 = \infty$. \square

4.3 The necessity of the transversality condition (15b)

Everywhere further, we assume the following condition holds:

Condition (∂) : for the maps $(t, x) \in \mathbf{T} \times \mathbf{X} \times \mathbf{U} \rightarrow f(t, x, u) \in \mathbf{X}$ and $(t, x) \in \mathbf{T} \times \mathbf{X} \times \mathbf{U} \rightarrow g(t, x, u) \in \mathbf{R}$ on their respective domains, there exist partial derivatives in x that are integrally bounded (on each compact) locally Lipschitz continuous Carathéodory maps.

Under this condition, the set $\partial_x \mathcal{H}(x(t), t, u^0(t), \lambda, \psi(t))$ is also a single-element set, therefore system (10a)–(10b) can be rewritten for $u = u^0$ in the form

$$\dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), t, u^0(t), \lambda, \psi(t)), \quad (24a)$$

$$\dot{x}(t) = f(t, x(t), u^0(t)), \quad (24b)$$

$$\dot{\lambda} = 0. \quad (24c)$$

Corollary 1 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b). If for each solution (ψ^0, x^0, λ^0) of system (24a)–(24c) with initial conditions from $K_0 \triangleq \mathbf{D} \times [0, 1] \times \{0_{\mathbf{X}}\}$ the component ψ^0 is partially Lyapunov stable in $G_0 \triangleq \mathbf{X} \times [0, 1] \times \mathbf{X}$.

Then, the result of Proposition 2 holds.

Proof. In (21), it is sufficient to define $E_p \triangleq \mathbf{X}$, $E_q \triangleq \mathbf{X} \times \mathbf{R}$ and to take ψ and (x, λ) for p and $q = (q_1, q_2)$, and to take for b and c the right-hand sides of (24a) and (24b)–(24c), respectively. Now Proposition 4 guarantees (ψ) , i.e., all conditions of Proposition 2 are met. This proves Corollary 1. \square

For G_0 , we can take the image $\{\varkappa(t, \psi(t), x(t), \lambda) \mid (x, u, \lambda, \psi) \in \tilde{\mathfrak{Y}}, t \in \mathbf{T}\}$.

Using Proposition 3 and Remark 4 instead of Proposition 4 and Corollary 1 in this proof, we obtain

Remark 6 Under conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$, for each uniformly overtaking optimal pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ for problem (1b), for each unbounded increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$, we have constructed the sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbf{N}} \in \tilde{\mathfrak{Y}}^{\mathbf{N}}$ such that

- 1) this sequence (as a sequence from $C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times \mathbf{T} \times C_{loc}(\mathbf{T}, \mathbf{X})$) converges to a certain $(x^0, \tilde{u}^0, \lambda^0, \psi^0) \in \mathfrak{Z}$;
- 2) the graphs $Gr(x^n, \lambda^n, \psi^n)$ of its elements are contained within the thinning funnels of solutions of system (24a)–(24c); i.e., for a sequence $(\delta_n)_{n \in \mathbf{N}} \in \mathbf{R}_{\geq 0}^{\mathbf{N}}$ that tends to 0, we have

$$\mathcal{K}(t, (\psi^n, x^n, \lambda^n)) \in (\psi^0(0), 0, \lambda^0) + \delta_n \mathbf{D} \times \delta_n \mathbf{D} \times [-\delta_n, \delta_n] \quad \forall t \in \mathbf{T}, n \in \mathbf{N}.$$

- 3) $\tilde{J}_{t_n}(\eta) - J_{t_n}(u^0) \rightarrow 0$, and $\psi^n(t_n) = 0$ for each $n \in \mathbf{N}$, where $(t_n)_{n \in \mathbf{N}}$ is a certain subsequence of $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$.

4.4 Modifications of transversality condition (15b)

In certain cases, if the Lagrange multiplier ψ is not stable, but we know that certain components of the vector variable ψ are stable, or we know the rate of its growth. Then we can try to select the mapping $A_* : \mathbf{T} \rightarrow \mathbf{L}$, may help to modify condition (15b), and use the condition

$$\liminf_{t \rightarrow \infty} \|\psi^0(t) A_*(t)\|_{\mathbf{X}} = 0 \quad (25a)$$

for certain map $A_* : \mathbf{T} \rightarrow \mathbf{L}$.

Here are the examples of such maps A_* : one that maps the unity matrix $A_*(t) \equiv \mathbf{1}_{\mathbf{L}}$; some “scalar” multiplier $A_*(t) \equiv r(t) \mathbf{1}_{\mathbf{L}}$; a mapping $A_*(\cdot) \equiv D$ with the diagonal matrix D ; the condition $\psi(t)x(t) \rightarrow 0$, which is often used as the sufficient condition, can also be reduced to this form.

Let us assume that for all $\eta \in \tilde{\mathfrak{U}}, \xi \in \mathbf{X}$, we have chosen the measurable mapping $A_\xi^\eta : \mathbf{T} \rightarrow \mathbf{L}$. Assume $A_\xi^\eta(0) = \mathbf{1}_{\mathbf{L}}$ for all $\eta \in \tilde{\mathfrak{U}}, \xi \in \mathbf{X}$. Define $A_* \triangleq A_{0\mathbf{X}}^{\tilde{u}^0}$.

Condition (ψA) : There exists a weight $w \in (Null)(u^0)$ such that for every solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ for each $\varepsilon \in \mathbf{R}_{>0}$ there exist a number $\delta \in \mathbf{R}_{>0}$ and a neighborhood $\mathcal{Y} \subset C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbf{T}, \mathbf{X})$ of the solution $(x^0, \tilde{u}^0, \lambda^0, \psi^0)$ such that for every solutions $z \triangleq (x, \eta, \lambda, \psi) \in \mathcal{Y} \cap \tilde{\mathfrak{Y}}$, from $\|\mathfrak{L}_w[\eta]\|_C < \delta$ it follows that $\|\psi A_{x(0)}^\eta - \psi^0 A_{0\mathbf{X}}^{\tilde{u}^0}\|_C < \varepsilon$.

Proposition 5 Assume conditions **(u)**, **(fg)** hold. For each uniformly overtaking optimal pair $(x^0, u^0) \in C(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ satisfying $(\psi \mathbf{A})$, for each unbounded increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ there exists $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ such that

$$\liminf_{n \rightarrow \infty} \|\psi^0(\tau_n) A_*(\tau_n)\|_{\mathbf{X}} = 0. \quad (25b)$$

hold.

The only differences between the proof of this Proposition and Proposition 2 are the facts that the references to (ψ) are replaced with references to (ψA) and the factors A_ξ^η, A_* are added to the inequalities of the last strings.

Similarly, we can formulate an analogue to Corollary 1 for this condition: if it is possible to choose matrix maps such that the product $\psi A_{z(0)}^u$ is the solution of an equation

$$\frac{dp}{dt} = b(t, p(t), \psi(t), x(t), \lambda, u(t)) \quad (26)$$

for each $u \in \mathfrak{U}$ for each solution $z = (\psi, x, \lambda)$ of system (10b),(10a),(24c) with initial conditions $z(0) \in E_q$, then the corresponding stability of this component p in system (26),(24a)–(24c) implies the result of Corollary 1 (see [24]).

The simplest way to account for the a priori information on stability or for asymptotic estimates of ψ and its components is to take $A_*(t) \triangleq e^{-\lambda t} \mathbf{1}_L$, where λ is greater than or equal to all Lyapunov's exponents of the variable ψ . In particular, in [28, Example 10.2], the use of $A_*(t) \triangleq e^{-t} \mathbf{1}_L$ in such condition (in contrast to the standard condition) selects the single extremal.

5 Cauchy formula for adjoint variable

In the papers [1, 2, 3, 4, 5], Aseev and Kryazhinskii have proposed and proven the analytic expression for the values of the adjoint variables. This version of the normal form of the Maximum Principle holds with the explicitly specified adjoint variable providing a complete set of necessary optimality conditions; moreover, the solution of this form of Maximum Principle is uniquely determined by the optimal control. This approach generalizes (see [4, Sect. 16], [7]) a number of transversality conditions; in particular, it is more general than the conditions that were obtained for linear systems in [8].

It turns out that if the function A_* is fundamental matrix of linearized system along the optimal trajectory, then, condition (25a) automatically yields this explicit representation for the adjoint variable.

Let us simplify Proposition 5 for such A_* to weaken the requirements of [4, Theorem 12.1],[7, Theorem 1],[5, Theorem 2], and their corollaries.

5.1 The case of dominating discount

Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b). Along with it, let us consider the solution A_* of the Cauchy problem

$$\frac{dA_*(t)}{dt} = \frac{\partial f(t, x^0(t), u^0(t))}{\partial x} A_*(t), \quad A_*(0) = \mathbf{1}_L.$$

Likewise, for each $\xi \in \mathbf{X}$, let us denote by x_ξ the solution of (10a) for the initial condition $x_\xi(0) = \xi \in \mathbf{X}$; let us also consider A_ξ , the solution of the matrix Cauchy problem

$$\frac{dA_\xi(t)}{dt} = \frac{\partial f(t, x_\xi(t), u^0(t))}{\partial x} A_\xi(t), \quad A_\xi(0) = \mathbf{1}_L \quad \forall \xi \in \mathbf{X}.$$

For each $T \in \mathbf{T}$, let us consider

$$I_\xi(T) \triangleq \int_0^T \frac{\partial g(t, x_\xi(t), u^0(t))}{\partial x} A_\xi(t) dt.$$

Proposition 6 *Assume conditions (\mathbf{u}) , (\mathbf{fg}) , (∂) . Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b). Let the map I_0 be bounded and let*

$$\lim_{\xi \rightarrow 0} \|I_\xi - I_0\|_C = 0.$$

Let $I_* \in \mathbf{X}$ be a partial limit (the limit of a subsequence) of $I_0(\tau)$ as $\tau \rightarrow \infty$.

Then, there exists a solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ of all relations of the Maximum Principle (10a)–(10d) satisfying the transversality condition (25a). Moreover, $\lambda^0 \triangleq \frac{1}{1 + \|I_*\|_{\mathbf{X}}} > 0$, and $\psi^0 \in C_{loc}(\mathbf{T}, \mathbf{X})$ defined by the following rule:

$$\psi^0(T) \triangleq \lambda^0 \left(I_* - \int_0^T \frac{\partial g(t, x^0(t), u^0(t))}{\partial x} A_*(t) dt \right) A_*^{-1}(T) \quad \forall T \in \mathbf{T}. \quad (27)$$

Proof. For each $u \in \mathfrak{U}$ and $z \triangleq \varphi[u]$, let us introduce a matrix function A^u that is the solution of the matrix equation

$$\dot{A}^u(t) = \frac{\partial f(t, z(t), u(t))}{\partial x} A^u(t), \quad A^u(0) = \mathbf{1}_{\mathbf{L}}. \quad (28)$$

Now, for each solution $(z, u, \lambda^u, \psi^u) \in \mathfrak{Y}$ from (11b) it follows that

$$\frac{d}{dt}(\psi^u A^u)(t) = -\lambda^u \frac{\partial g(t, z(t), u(t))}{\partial x} A^u(t). \quad (29)$$

For

$$\begin{aligned} E_p &\triangleq \mathbf{X}, \quad b(t, p, (q_1, q_2, q_3), u) \triangleq -q_3 \frac{\partial g(t, q_2, u)}{\partial x} q_1, \\ E_q &\triangleq \mathbf{L} \times \mathbf{X} \times \mathbf{R}, \quad c(t, p, (q_1, q_2, q_3), u) \triangleq \left(\frac{\partial f(t, q_2, u)}{\partial x} q_1, f(t, q_2, u), 0 \right) \end{aligned} \quad (30)$$

the system (21) becomes system (29), (28), (10a), (24c); now, for $u = u^0$,

$$\dot{p} = -r \frac{\partial g(t, z, u^0(t))}{\partial x} B, \dot{B} = \frac{\partial f(t, z, u^0(t))}{\partial x} B, \dot{z} = f(t, z, u^0(t)), \dot{r} = 0. \quad (31)$$

Solving this system, we obtain

$$\begin{aligned} r(t) &= r(0), \quad z(t) = x_{z(0)}(t), \quad B(t) = A_{z(0)}(t)B(0), \\ p(t) &= p(0) - r(0)I_{z(0)}(t)B(0). \end{aligned} \quad (32)$$

Let $G_0 \triangleq \mathbf{X} \times \mathbf{L} \times \mathbf{X} \times [0, 1]$. We claim the partial Lyapunov stability of the component p in G_0 for $B(0) = \mathbf{1}_{\mathbf{L}}, z(0) = 0, r(0) \in [0, 1], p(0) \in \mathbf{D}$. Indeed, by (32) it remains to verify that I_ξ is continuous at the point $\xi = 0$, and I_0 is bounded; both hold by assumptions. Therefore, by Proposition 4, for the certain weight $w^0 \in (Null)(u^0)$, the component p is stable for \mathfrak{L}_{w^0} –small perturbations of the control u^0 . This provides condition (ψA) for A^u that were defined as we did.

By condition, I_* is a partial limit; hence, there exists an unbounded increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ with property $I_0(\tau_n) \rightarrow I_*$. Now, by Proposition 5, there exists $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ with properties (25b) and (25a).

Substituting $z(0) = 0, r(0) = \lambda^0, A_0(0) = \mathbf{1}_{\mathbf{L}}$, and $p(0) = \psi^0(0)$ into (32) yields

$$(\psi^0 A_*)(T) = (\psi^0 A_0)(T) = p(T) = \psi^0(0) - \lambda^0 I_0(T) \quad \forall T \in \mathbf{T}. \quad (33)$$

Now, substituting $T = \tau_n$ and passing to the lower limit, from (25a), we obtain $0 = \psi^0(0) - \lambda^0 I_*$; therefore, from (33) and (10d) respectively, we have

$$\psi^0(T)A_*(T) = \lambda^0(I_* - I_0(T)), \quad \lambda^0 = \frac{1}{1 + \|I_*\|_{\mathbf{X}}} > 0.$$

Using the inverse matrix for $A_*(T)$, we obtain (27). \square

Let us note that if I_* is independent of choice of the subsequence $(\tau_n)_{n \in \mathbf{N}}$, we automatically obtain the stronger transversality condition

$$\lim_{t \rightarrow \infty} \psi(t) A_*(t) = 0. \quad (34)$$

Moreover, since for different (x^0, u^0, λ) , solutions of (29) differ by a constant, for all $(x^0, u^0, \lambda, \psi) \in \mathfrak{Y}$, the products ψA_0 tend to a finite limit as $t \rightarrow \infty$. If (27) holds, then this limit is equal to zero. Hence, to every (x^0, u^0, λ) there corresponds at most one ψ^0 , for which relations (10a)–(10c), (34) hold; now, from (10d) and (32) we can reconstruct λ^0 uniquely. Thus there exists the unique solution $(x^0, u^0, \lambda, \psi) \in \mathfrak{Z}$ that satisfies condition (34), and the following theorem is proved.

Theorem 2 *Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b), and let the limit*

$$\lim_{t \rightarrow \infty, \xi \rightarrow 0_{\mathbf{X}}} I_{\xi}(t) = \int_{\mathbf{T}} \frac{\partial g(t, x^0(t), u^0(t))}{\partial x} A_*(t) dt \in \mathbf{R}$$

be well-defined and finite.

Then, there exists the unique solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ of all relations of the Maximum Principle (10a)–(10d) satisfying the transversality condition (34). Moreover, accurate to the positive factor, we can assume

$$\lambda^0 \triangleq 1, \quad \psi^0(T) \triangleq \int_{[T, \infty)} \frac{\partial g(t, x^0(t), u^0(t))}{\partial x} A_*(t) dt A_*^{-1}(T) \quad \forall T \in \mathbf{T}. \quad (35)$$

From conditions of [5, Theorem 2], [4, Theorem 12.1], [3, Theorem 1], [7, Theorem 1] it follows that for some $\alpha, \beta \in \mathbf{R}_{>0}$ and for all admissible controls u , all trajectories x , and all fundamental matrices A^u , the inequality

$$\left\| \frac{\partial g(t, x(t), u(t))}{\partial x} \right\| \|A^u(t)\| \leq \beta e^{-\alpha t} \quad \forall t \in \mathbf{T} \quad (36)$$

holds. This is stronger than the conditions of Theorem 2. Informally, the requirements of [3, Theorem 1], [5, Theorem 2], [4, Theorem 12.1], [7, Theorem 1] boil down to the need for uniform exponential Lyapunov stability of the product ψA along all trajectories of the system (1), while Lyapunov stability of the product $\psi^0 A_0$ along the optimal solution of the initial control problem is sufficient for Theorem 2. On the other side, the condition (36) can be verified by calculating the Lyapunov exponents of the system of the Maximum Principle, see [4, Sect. 12], [5, Sect. 3], [7, Sect. 5].

5.2 The general case

Let us base on the start of the proof of Proposition 6, and let us use not Corollary 1 but Remark 4.

Proposition 7 *Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b).*

Now, for an unbounded increasing sequence of times $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$, there exist:

- 1) its subsequence $(t_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$;
- 2) the sequence of initial conditions $(\zeta_n)_{n \in \mathbf{N}} \in \mathbf{X}^{\mathbf{N}}$ that converges to $0_{\mathbf{X}}$;
- 3) the sequence $(\lambda_n)_{n \in \mathbf{N}} \in [0, 1]^{\mathbf{N}}$ that converges to some $\lambda^0 \in [0, 1]$;

such that if $\psi^0 \in C(\mathbf{T}, \mathbf{X})$ is defined for every $t \in \mathbf{T}$ by the rule

$$\psi^0(T) = \lim_{n \rightarrow \infty} \lambda_n \int_T^{t_n} \frac{\partial g(t, x_{\zeta_n}(t), u^0(t))}{\partial x} A_{\zeta_n}(t) dt A_*^{-1}(T),$$

then, the limit would be uniform on every compact, and $(x^0, u^0, \lambda^0, \psi^0)_{n \in \mathbf{N}} \in \mathfrak{Z}$ would satisfy all relations of the Maximum Principle (10a)–(10d). Moreover,

$$\psi^0(T) = \lim_{n \rightarrow \infty} \lambda^n \int_T^{t_n} \frac{\partial g(t, x_{\zeta_n}(t), u^0(t))}{\partial x} A_{\zeta_n}(t) dt A_{\zeta_n}^{-1}(T) \quad \forall T \in \mathbf{T}. \quad (37)$$

Proof. Indeed, consider the control system $(\dot{p}, \dot{q}) = (b, c) = a$ from (30). It features the set of controls \mathfrak{U} , however, as a system of form (3), it defines the control system of form (4) that is controlled by the elements of \mathfrak{U} . For such a system, fix the weight w^0 from the formulation of Proposition 3.

By Remark 4, for every $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$, there exist its subsequence $(t_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ and the sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbf{N}} \in \tilde{\mathfrak{Y}}$ of solutions of system (10a)–(10b), converging to the certain solution $(x^0, \tilde{u}^0, \lambda^0, \psi^0) \in \tilde{\mathfrak{Z}}$ of all relations of the Maximum Principle.

Now, for every $n \in \mathbf{N}$, we can find $B_n \in C(\mathbf{T}, \mathbf{L})$ and $p_n \in C(\mathbf{T}, \mathbf{X})$ such that

$$a^n \triangleq (p_n, B_n, x^n, \lambda^n) \in \tilde{\mathfrak{A}}[\eta^n], \quad B_n(0) = 1_{\mathbf{L}}, \quad p_n(0) = \psi^n(0). \quad (38)$$

On the other side, differentiating $\psi^n B_n$ (as in (29)), we check that $(\psi^n B_n, B_n, x^n, \lambda^n) \in \tilde{\mathfrak{A}}[\eta^n]$. Comparing the initial conditions, we see that $p_n \equiv \psi^n B_n$.

For each $n \in \mathbf{N}$, for each $t \in \mathbf{T}$, there exists the solution $a_{n,t} \in C(\mathbf{T}, E)$ of (31) for the initial conditions $a_{n,t}(0) = \varkappa(t, a^n(t))$. Note that the last components of $a_{n,t}$ and a^n are independent of t ; thus, they correspond with λ^n . Now we can correctly define the components of the map $t \mapsto \varkappa(t, a^n(t))$ by the rule

$$\varkappa(t, a^n(t)) = (\nu_n(t), \mu_n(t), \xi^n(t), \lambda^n) \quad \forall t \in \mathbf{T}, n \in \mathbf{N}.$$

Substituting these initial conditions into (32), by virtue of equalities (38) and $a^n(t) = a_{n,t}(t)$ for all $n \in \mathbf{N}, t \in \mathbf{T}$, we obtain

$$(\psi^n(t) B_n(t), B_n(t), x^n(t)) = a_{n,t}(t) = (\nu_n(t) - \lambda^n I_{\xi^n(t)} \mu_n(t), A_{\xi^n(t)} \mu_n(t), x_{\xi^n(t)}).$$

Specifically,

$$\psi^n(t) A_{\xi^n(t)}(t) = \psi^n(t) B^n(t) \mu_n^{-1}(t) = \nu_n(t) \mu_n^{-1}(t) - \lambda^n I_{\xi^n(t)}(t).$$

Remark 4 provides $\psi^n(t_n) = 0$; substituting $t = t_n$, we obtain

$$0 = \psi^n(t_n) A_{\xi^n(t_n)}(t_n) = \nu_n(t_n) \mu_n^{-1}(t_n) - \lambda^n(t_n) I_{\xi^n(t_n)}(t_n).$$

Subtracting one from another yields

$$\psi^n(t) A_{\xi^n(t)}(t) = \nu_n(t) \mu_n^{-1}(t) - \lambda^n I_{\xi^n(t)}(t) - \nu_n(t_n) \mu_n^{-1}(t_n) + \lambda^n I_{\xi^n(t_n)}(t_n). \quad (39)$$

By Remark 4, we have $a^n(0) \stackrel{(38)}{=} (\psi^n(0), 1_{\mathbf{L}}, 0_{\mathbf{X}}, \lambda^n) \rightarrow (\psi^0(0), 1_{\mathbf{L}}, 0_{\mathbf{X}}, \lambda^0)$, and $\|\mathfrak{L}_{w^0}(\eta^n)\|_C \rightarrow 0$ as $n \rightarrow \infty$; moreover, Proposition 3 yields for all $t \in \mathbf{T}$

$$\max_{t \in \mathbf{T}} \|a_{t,n}(0) - a^n(0)\|_E = \max_{t \in \mathbf{T}} \|\mathfrak{z}(t, a^n(t)) - \mathfrak{z}(0, a^n(0))\|_E \leq \|\mathfrak{L}_{w^0}(\eta^n)\|_C \rightarrow 0.$$

Hence uniformly on the whole \mathbf{T} as $n \rightarrow \infty$, it holds that

$$a_{t,n}(0) = (\nu_n(t), \mu_n(t), \xi^n(t), \lambda^n) \rightarrow (\psi^0(0), 1_{\mathbf{L}}, 0_{\mathbf{X}}, \lambda^0) \quad \forall t \in T. \quad (40)$$

Whence the theorem of continuous dependence on initial conditions yields the uniformity of the limits

$$\lim_{n \rightarrow \infty} \nu_n(\tau) \mu_n^{-1}(\tau) = \psi^0(0), \quad \lim_{n \rightarrow \infty} \lambda^n I_{\xi^n(\tau)}(t) = \lambda^0 I_0(t) \quad \forall t \in K, \tau \in \mathbf{T}.$$

as $n \rightarrow \infty$ for each compact $K \in (\text{comp})(\mathbf{T})$. Putting here $\tau = t$, $\tau = t_n$, let us consider the limit of both sides of (39) as $n \rightarrow \infty$; thus,

$$\begin{aligned} \psi^0(t) A_0(t) &= \lim_{n \rightarrow \infty} \left(\psi^0(0) - \lambda^0 I_0(t) - \psi^0(0) + \lambda^n I_{\xi^n(t_n)}(t_n) \right) = \\ \lim_{n \rightarrow \infty} \left(-\lambda^n I_{\xi^n(t_n)}(t) + \lambda^n I_{\xi^n(t_n)}(t_n) \right) &= \lim_{n \rightarrow \infty} \lambda^n (I_{\xi^n(t_n)}(t_n) - I_{\xi^n(t_n)}(t)). \end{aligned}$$

Multiplying on the right by $A_0^{-1}(t) = A_*^{-1}(t)$ and $A_0(t) A_{\xi^n(t_n)}(t)$ we obtain our proposition for $\zeta_n \triangleq \xi^n(t_n)$. All necessary convergences are provided by uniformity of limits in (40). \square

We say an optimal pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ for problem (1a)–(1b) is *abnormal* if every solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ of all relations of the Maximum Principle (10a)–(10d) satisfies $\lambda^0 = 0$.

Remark 7 Assume conditions (\mathbf{u}) , (\mathbf{fg}) , (∂) . Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal pair for problem (1a)–(1b) and let this pair be abnormal. Then,

$$\limsup_{\tau \rightarrow \infty, \xi \rightarrow 0} \|I_\xi(\tau)\|_E = \limsup_{\tau \rightarrow \infty, \xi \rightarrow 0} \left\| \int_0^\tau \frac{\partial g(t, x_\xi(t), u^0(t))}{\partial x} A_\xi(t) dt \right\|_E = \infty.$$

Indeed, if it is wrong, then, the right-hand side of (37) equals zero for $T = 0$, i.e., $\psi^0(0) = 0_{\mathbf{X}}$, which contradicts the relation (10d) for $\lambda^0 = 0$.

5.3 Monotonous case

Consider the case of when both the right-hand side of the dynamics equation and the objective function are monotonous. This case frequently arises in economical applications while monotonicity simplifies its examination. It seems that the first to note the peculiarities of this case and to investigate it were Aseev, Kryazhinskii, and Taras'ev in their paper [6]. These were followed by papers [35], [1], [2], and the most general case was considered in [4].

In Euclidean space E' , let us define binary relations \succsim and \succ by the rules

$$(\alpha \succsim \beta) \Leftrightarrow (\alpha - \beta \in \mathbf{T}^{dim E}), \quad (\alpha \succ \beta) \Leftrightarrow (\alpha - \beta \in \mathbf{R}_{>0}^{dim E}) \quad \forall \alpha, \beta \in E'.$$

This allows us to use the symbols \succsim and \succ to compare vectors and matrices, and vector and matrix functions. For the latter two, \succsim and \succ allow us to discuss their monotonicity.

Proposition 8 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b). Assume for all $x \in \mathbf{X}$ and for a.a. $t \in \mathbf{T}$ there exists a number $d(t, x) \in \mathbf{R}$ such that the following relation holds:

$$\frac{\partial g(t, x, u^0(t))}{\partial x} \succcurlyeq 0_{\mathbf{L}}, \quad \frac{\partial f(t, x, u^0(t))}{\partial x} \succcurlyeq d(t, x)1_{\mathbf{L}}.$$

Then, there exists a solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ of all relations of the Maximum Principle (10a)–(10d) satisfying (37), and $\psi^0 \succcurlyeq 0_{\mathbf{X}}$.

If at the same time the pair (x^0, u^0) is normal, then

$$\lambda^0 \limsup_{t \rightarrow \infty, \xi \rightarrow 0} I_{\xi}(t) \succcurlyeq \psi^0(0) \succcurlyeq \lambda^0 \lim_{t \rightarrow \infty} I_0(t) \succcurlyeq 0_{\mathbf{X}} \quad (41)$$

hold, and all limits in (41) well-defined and finite.

Corollary 2 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)–(1b), and let this pair be normal. Assume for all $x \in \mathbf{X}$ and for a.a. $t \in \mathbf{T}$ there exists a number $d(t, x) \in \mathbf{R}$ such that the following relation holds:

$$\frac{\partial g(t, x, u^0(t))}{\partial x} \succ 0_{\mathbf{L}}, \quad \frac{\partial f(t, x, u^0(t))}{\partial x} \succ d(t, x)1_{\mathbf{L}}.$$

Then, there exists a solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ of all relations of the Maximum Principle (10a)–(10d) satisfying (37), (41), and $\psi^0 \succ 0_{\mathbf{X}}$.

Proof. Below, in the proof of Proposition 8, we understand the symbol \triangleright as \succcurlyeq , and in the proof of Corollary 2, we understand it as \succ .

Fix arbitrary $\xi \in \mathbf{X}, T \in \mathbf{R}_{>0}, \tau \in \langle T, \infty \rangle$; let us show that $A_{\xi}(\tau)A_{\xi}^{-1}(T) \triangleright 0_{\mathbf{L}}$. Denote by $F_{\xi}(t)$ the matrix $\frac{\partial f(t, x_{\xi}(t), u^0(t))}{\partial x}$ for all $t \in [T, \tau]$. The diagonal of the map F_{ξ} is dominated by a function $M \triangleq M_{[T, \tau]}^{F_{\xi}} \in \mathcal{L}_{loc}^1(\mathbf{T}, \mathbf{T})$; then, by condition, $F_{\xi} + m(t)1_{\mathbf{L}}|_{\langle T, \tau \rangle} \triangleright 0_{\mathbf{L}}$. Now, let us consider a solution $P(t)$ of the equation

$$\dot{P} = (F_{\xi}(t) + M(t)1_{\mathbf{L}})P, P(T) = 1_{\mathbf{L}}, t \geq T;$$

for it, it holds that $P(t) \triangleright 0_{\mathbf{L}}$ for all $t \in \langle T, \tau \rangle$. Since A_{ξ} and $1_{\mathbf{L}}$ commute, the solution P is the product of two solutions of the equations $\dot{Q} = F_{\xi}(t)Q, Q(T) = 1_{\mathbf{L}}$, and $\dot{R} = M(t)1_{\mathbf{L}}R, R(T) = 1_{\mathbf{L}}$. Thus,

$$P(\tau) = Q(\tau)R(\tau) = Q(\tau)e^{\int_T^{\tau} M(t)dt}1_{\mathbf{L}} = A_{\xi}(\tau)A_{\xi}^{-1}(T)e^{\int_T^{\tau} M(t)dt},$$

and $P(\tau) \triangleright 0_{\mathbf{L}}$ implies $A_{\xi}(\tau)A_{\xi}^{-1}(T) \triangleright 0_{\mathbf{L}}$ for all $\tau \in \langle T, \infty \rangle$. Now, by monotonicity of matrix product, we obtain

$$\frac{dI_{\xi}(t)}{dt}A_{\xi}^{-1}(T) = \frac{\partial g(t, x_{\xi}(t), u^0(t))}{\partial x}A_{\xi}(t)A_{\xi}^{-1}(T) \triangleright 0_{\mathbf{X}} \quad \forall t \in \langle T, \infty \rangle \quad (42)$$

for all $\xi \in \mathbf{X}, T \in \mathbf{T}$; specifically, for $T = 0$ we have $\frac{dI_{\xi}(t)}{dt} \triangleright 0_{\mathbf{X}}$, hence the functions $I_{\xi}, I_{\xi}A_{\xi}^{-1}(T)$ are monotonically increasing for all $\xi \in \mathbf{X}, T \in \mathbf{T}$.

By Proposition 7, there exists the solution $(x^0, u^0, \psi^0, \lambda^0)$ of relations of the Maximum Principle satisfying of formula (37) for certain sequences λ^n and ξ_n . However, the expression into the limit of (37) lies in $\mathbf{L}_{\geq 0}$ by (42). Passing to the limit as $n \rightarrow \infty$, we obtain $\psi^0 \succcurlyeq 0_{\mathbf{X}}$.

Suppose the pair (x^0, u^0) is normal; then $\lambda^0 > 0$. Since the function I_ξ is monotonically increasing and, by Remark 7, uniformly bounded in the certain neighborhood $0_{\mathbf{X}}$, the Lebesgue theorem yields the existence of the finite limits in (41). Hence,

$$\lambda_0 \limsup_{t \rightarrow \infty, \xi \rightarrow 0} I_\xi(t) \succcurlyeq \lambda_0 \lim_{n \rightarrow \infty} I_{\xi_n}(t_n) \stackrel{(37)}{=} \psi^0(0).$$

On the other side, monotonicity of $I_\xi A_\xi^{-1}(T)$ yields

$$\begin{aligned} \frac{1}{\lambda^0} \psi^0(T) &\stackrel{(37)}{=} \lim_{n \rightarrow \infty} (I_{\xi_n}(t_n) - I_{\xi_n}(T)) A_{\xi_n}^{-1}(T) \stackrel{(42)}{\triangleright} \lim_{n \rightarrow \infty} (I_{\xi_n}(t) - I_{\xi_n}(T)) A_{\xi_n}^{-1}(T) \\ &= (I_0(t) - I_0(T)) A_0^{-1}(T) \stackrel{(42)}{\triangleright} 0_{\mathbf{X}} \quad \forall T \in \mathbf{T}, t \in \langle T, \infty \rangle, \end{aligned}$$

i.e. $\psi^0 \triangleright 0_{\mathbf{X}}$. Moreover, substituting $T = 0$ and passing to the limit as $t \rightarrow \infty$, we obtain the lower estimate from (41). \square

Note that in [6, Theorem 1], [4, Theorem 10.1] the estimate $\psi \succcurlyeq 0_{\mathbf{X}}$ ($\psi \succ 0_{\mathbf{X}}$) is made for autonomous systems under less general assumptions; in [4, Theorem 10.1], the lower estimate from (41) was made too (see [4, (10.17)]). However, in these papers, the condition $\lambda > 0$ was not assumed but proved; namely, with the aid of the normal-form stationarity condition, the boundedness of integrals of (37) was proved, which guaranteed the control was normal.

Let us also note that formula (35) was also proved for biaffine control system for monotonic $\frac{\partial q}{\partial x}$ ([3, Theorem 1], [4, Theorem 11.1]). It seems, this result is not a direct consequence of Theorem 2, which was proved in that paper.

6 Addendum

In the paper, the left endpoint is considered to be fixed. It seems this condition may be easily discarded, since to do it, it is sufficient to equip the finite horizon optimization problems from the proof of Theorem 1 with the same condition for the left endpoint and to provide the boundedness of $x(0)$.

6.1 Case of σ -compact-valued map U

The condition (u) implies that at every time $t \in \mathbf{T}$, the controls are chosen from the compact $U(t)$. Let us weaken this assumption to the following:

Condition (u_σ) : U is a σ -compact-valued map such that $Gr U \in \mathcal{B}(\mathbf{T} \times \mathbf{U})$.

We shall still assume the conditions (a), (fg) to hold, and we shall not change the definition of \mathfrak{U} . Then, we can assume there exists a nondecreasing sequence $(U^{(r)})_{r \in \mathbf{N}}$ of integrally bounded (on each compact subset of \mathbf{T}) compact-valued maps such that $U \equiv \cup_{r \in \mathbf{N}} U^{(r)}$. Let us assume the uniformly overtaking optimal control u^0 exists. Then, we may safely assume that $Gr u^0 \subset Gr U^{(1)}$.

Repeating the reasonings of Sect 2, for each $r \in \mathbf{N}$, we can construct sets $\mathfrak{U}^{(r)}, \tilde{\mathfrak{U}}^{(r)}$ and their images for the restriction: $\mathfrak{U}_n^{(r)} \triangleq \pi_n(\mathfrak{U}^{(r)}), \tilde{\mathfrak{U}}_n^{(r)} \triangleq \tilde{\pi}_n(\tilde{\mathfrak{U}}^{(r)})$. Let us introduce the set $\tilde{\mathfrak{U}}$ of all maps η from \mathbf{T} into the set of Radon probability measures over \mathbf{U} such that for every $n \in \mathbf{N}$ there exists $r = r(\eta, n) \in \mathbf{N}$ such that $\pi_n(\eta) = \eta|_{[0, n]} \in \tilde{\mathfrak{U}}_n^{(r)}$. The topology of this set is of no use to us, thus we assume it is indiscrete. Note that under our definition, $\tilde{\delta}(\mathfrak{U}) \not\subset \tilde{\mathfrak{U}}$, but $u^0 \in \tilde{\delta}(\mathfrak{U}^{(r)}) \subset \tilde{\delta}(\mathfrak{U})$ for all $r \in \mathbf{N}$.

Note that, for all $\eta \in \tilde{\mathfrak{U}}$, the set $\tilde{\mathfrak{A}}[\eta]$ is still compact. Indeed, for each $n \in \mathbf{N}$, it holds that $\tilde{\mathfrak{A}}[\eta]|_{[0, n]} \subset \tilde{\mathfrak{A}}[\tilde{\mathfrak{U}}^{(r(\eta, n))}]|_{[0, n]} \in (comp)(C([0, n], E))$; all that remains is to use the definition of compact-open topology. For each $r \in \mathbf{N}$, denote by $Z^{(r)}$ the pairs $(\psi, \lambda) \in C(\mathbf{T}, \mathbf{X})$ such that $(x^0, u^0, \psi^r, \lambda^r)$ satisfies relations (10a)–(10b), (10d), and for a.a. $t \in \mathbf{T}$ instead of (10c), there holds the weaker relation

$$\sup_{p \in U^{(r)}(t)} \mathcal{H}(x(t), t, p, \lambda, \psi(t)) = \mathcal{H}(x(t), t, u^0(t), \lambda, \psi(t)). \quad (43)$$

Note that this set is compact (it follows from compactness of $\tilde{\mathfrak{A}}[\eta]$). By Theorem 1, $Z^{(r)}$ is not empty for all $r \in \mathbf{N}$. It is easily seen that $Z^{(r')} \subset Z^{(r'')}$ for any $r', r'' \in \mathbf{N}, (r' \subset r'')$. Then, there exists $(\psi^0, \lambda^0) \in \cap_{r \in \mathbf{N}} Z^{(r)}$. Therefore, for it, (43) holds for all $r \in \mathbf{N}$; thus, (10c) holds too; hence, $(x^0, u^0, \psi^0, \lambda^0)$ satisfies all relations (10a)–(10d) of the Maximum Principle.

For each $r \in \mathbf{N}$, consider the sequences $(x_r^n, \eta_r^n, \lambda_r^n, \psi_r^n)_{n \in \mathbf{N}}, (t_{n,r})_{n \in \mathbf{N}}$ from Remark 4. Then, for the sequence $(x_n^n, \eta_n^n, \lambda_n^n, \psi_n^n)_{n \in \mathbf{N}}$, by uniformity of estimate (14), we have pointwise convergence of η_n to \tilde{u}_0 ; moreover, for each $k \in \mathbf{N}$ in the interval $[0, k]$ for this sequence, the convergences from Remark 4 hold (it is sufficient to consider there topologies with respect to $C([0, k], \mathbf{X}), \tilde{\mathfrak{U}}^{(k)}$), specifically, it would hold that

- 1) $(x^n, \eta^n, \lambda_n^n, \psi_n^n) \rightarrow (x^0, \tilde{u}^0, \lambda^0, \psi^0)_{n \in \mathbf{N}} \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \tilde{\mathfrak{U}} \times \mathbf{T} \times C_{loc}(\mathbf{T}, \mathbf{X})$;
- 2) $\|\mathfrak{L}_w(\eta^n)\|_C \rightarrow 0$;
- 3) $\tilde{J}_{t_n}(\eta) - J_{t_n}(u^0) \rightarrow 0$, and $\psi_n^n(t_n) = 0$ for each $n \in \mathbf{N}$, where $t_n \triangleq t_{n,n}$.

The verbatim repetition of the proof of Proposition 2 yields

Proposition 9 *Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg})$. For each uniformly overtaking optimal pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ satisfying (ψ) for each unbounded increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ there exists $(x^0, u^0, \lambda^0, \psi^0)$ such that the relations of the Maximum Principle (10a)–(10d), and the transversality condition (15b) hold.*

Since our case is more general, starting with Sect. 4, the references to (\mathbf{u}) ought to be replaced with (\mathbf{u}_σ) , and the results of Sect. 3 ought to be replaced with their respectful analogues.

6.2 On uniformly sporadically catching up controls.

Definition 3 We say that a control $u^0 \in \mathfrak{U}$ is uniformly sporadically catching up optimal if for every $\varepsilon, T \in \mathbf{R}_{>0}$ there exists $t \in [T, \infty)$ such that $J_t(u^0) \geq J_t(u) - \varepsilon$ holds for all $u \in \mathfrak{U}$.

Note that for each uniformly sporadically catching up optimal control, there exists an unbounded monotonically increasing sequence $(\tau_n)_{n \in \mathbf{N}} \in \mathbf{T}^{\mathbf{N}}$ and a function $\omega^0 \in \Omega$ such that

$$J_{\tau_n}(u^0) \geq J_{\tau_n}(u) - \omega^0(\tau_k) \quad \forall u \in \mathfrak{U}, k, n \in \mathbf{N}, k < n.$$

We call such control a τ -sporadically catching up optimal.

Now, if we consider the sequence $(\tau_n)_{n \in \mathbf{N}}$ everywhere defined and understand the optimality in the above sense, then all statements, starting with Theorem 1, hold. In particular, we can rewrite Proposition 6 and Theorem 2 in the following way:

Theorem 3 Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\partial)$ hold. Let the pair $(x^0, u^0) \in C_{loc}(\mathbf{T}, \mathbf{X}) \times \mathfrak{U}$ be τ -sporadically catching up optimal for problem (1a)–(1b). Let $I_* \in \mathbf{X}$ be the limit of $I_\xi(\tau_n)$ as $n \rightarrow \infty, \xi \rightarrow 0_{\mathbf{X}}$.

Then, there exists the unique solution $(x^0, u^0, \lambda^0, \psi^0)$ of all relations of the Maximum Principle (10a)–(10d) and the transversality condition (25b). Moreover, accurate to the positive factor, we can assume

$$\lambda^0 \triangleq 1, \quad \psi^0(T) \triangleq \left(I_* - \int_0^T \frac{\partial g(t, x^0(t), u^0(t))}{\partial x} A_*(t) dt \right) A_*^{-1}(T) \quad \forall T \in \mathbf{T}.$$

Acknowledgements I would like to express my gratitude to A. G. Chentsov, A. M. Tarasyev, N. Yu. Lukoyanov, and Yu. V. Averboukh for valuable discussion in course of writing this article. Special thanks to Ya. Sali for the translation.

References

1. S. M. Aseev, A. V. Kryazhinskii, The Pontryagin Maximum Principle for Infinite-Horizon Optimal Controls, IIASA Interim Rept. IR-03-013 (Laxenburg, Austria), 2003.
2. S. M. Aseev, A. V. Kryazhinskii, The Pontryagin Maximum Principle and transversality conditions for a class of optimal control problems with infinite time horizons, SIAM J. Control Optim. 43, 1094-1119, 2004.
3. S. M. Aseev, A. V. Kryazhinskii, The Pontryagin Maximum Principle for an Optimal Control Problem with a Functional Specified by an Improper Integral, Dokl. Math. 69, 1, 89-91, 2004.
4. S. M. Aseev, A. V. Kryazhinskii, The Pontryagin Maximum Principle and problems of optimal economic growth, Proc. Steklov Inst. Math. 257, no. 1, 1-255, 2007.
5. S. M. Aseev, A. V. Kryazhinskii, Shadow prices in infinite-horizon optimal control problems with dominating discounts, Applied Mathematics and Computation V.204, 2, 519-531, 2008.
6. S. M. Aseev, A. V. Kryazhinskii, A. M. Tarasyev, The Pontryagin Maximum Principle and Transversality Conditions for an Optimal Control Problem with Infinite Time Interval, Proc. Steklov Inst. Math. 233, 64-80, 2001.
7. S. M. Aseev, V. M. Veliov, Maximum Principle for infinite-horizon optimal control; problems with dominating discount, Dynamics of Continuous, Discrete and Impulsive Systems (in appear?)
8. J. P. Aubin, F. H. Clarke, Shadow Prices and Duality for a Class of Optimal Control Problems, SIAM J. Control Optim. 17, 567-586, 1979.
9. E. J. Balder, An existence result for optimal economic growth problems, J. of Math. Anal. 95, 1, 195-213, 1983.
10. L. M. Benveniste, J. A. Scheinkman, Duality theory for dynamic optimization models of economics: The continuous time case, Journal of Economic Theory, 27, 1-19, 1982.

11. *D. A. Carlson*, Uniformly overtaking and weakly overtaking optimal solutions in infinite-horizon optimal control: when optimal solutions are agreeable, *J. Optim. Theory Appl.*, 64, 1, 55-69, 1990.
12. *D. A. Carlson*, Nonconvex and Relaxed Infinite-Horizon Optimal Control Problems, *J. Optim. Theory Appl.* 78, 3, 465-491, 1993.
13. *D. A. Carlson, A. B. Haurie, A. Leizarowitz*, Infinite Horizon Optimal Control. Deterministic and Stochastic Systems. Springer, Berlin, 1991.
14. *F. H. Clarke*, Optimization and Nonsmooth Analysis. Wiley, New York, 1983.
15. *H. W. Daniel*, Survey of measurable selection theorems: an update // *Lect. Notes Math.*, 794, Springer, 176-219, 1980.
16. *A. V. Dmitruk, N. V. Kuz'kina*, An existence theorem in an optimal control problem on an infinite time interval, *Math. Notes* 78, 3-4, 466-480, 2005.
17. *R. Engelking*, General Topology, PWN, Warszawa, 1983.
18. *V. V. Fedorchuk, V. V. Filippov*, General topology. Basic Constructions. (in Russian) Fizmatlit, Moscow, 2006.
19. *R. V. Gamkrelidze*, Principles of Optimal Control Theory. Plenum Press, New York, 1978.
20. *H. Halkin*, Necessary Conditions for Optimal Control Problems with Infinite Horizons, *Econometrica* 42, 267-272, 1974.
21. *P. J. Hammond, J. Kennan*, Uniformly optimal infinite-horizon plans, *International Economic Review*, 20, 283-296, 1979.
22. *T. Kamihigashi*, Necessity of transversality conditions for infinite horizon problems, *Econometrica*, 69, 995-1012, 2001.
23. *D. V. Khlopin* On extension of conflict control problems on infinite horizon (In Russian), *Vestn. Udmurtsk. Univ. Mat. Mekh. Comp. Nauki*, No 1, 105-112, 2011.
24. *D. V. Khlopin* Necessary conditions of optimality in the case of stable adjoint variable for infinite horizon problems with free right endpoint (In Russian), *Vestn. Udmurtsk. Univ. Mat. Mekh. Comp. Nauki*, (in appear?) 2011
25. *P. Michel* On the transversality condition in infinite horizon optimal problems, *Econometrica*, 50, 975-984, 1982.
26. *L. S. Pontryagin et al.*, The Mathematical Theory of Optimal Processes, Fizmatgiz, Moscow, 1961.
27. *N. Sagara*, Value functions and transversality conditions for infinite-horizon optimal control problems // *Set-Valued Var. Anal.* 18, 1, 1-28, 2010.
28. *A. Seierstad*, Necessary conditions for nonsmooth, infinite-horizon optimal control problems, *J. Optim. Theory Appl.*, 103, 1, 201-230, 1999.
29. *A. Seierstad, K. Sydsæter*, Conditions implying the vanishing of the Hamiltonian at infinity in optimal control problems, *Optim. Lett.*, 3, 507-512, 2009.
30. *K. Shell*, Applications of Pontryagin's maximum principle to economics, *Mathematical Systems Theory and Economics* 1, *Lect. Notes Oper. Res. Math. Econ.* vol. 11, Springer, Berlin, 241-292, 1969.
31. *G. V. Smirnov*, Transversality condition for infinite-horizon problems, *J. Optim. Theory Appl.* 88, 3, 671-688, 1996.
32. *L. E. Stern*, Criteria of optimality in the infinite-time optimal control problem, *J. Optim. Theory Appl.* 44, 3, 497-508, 1984.
33. *A. A. Tolstonogov*, Differential inclusions in a Banach space. Mathematics and its Applications, 524. Kluwer Academic Publishers, Dordrecht, 2000.
34. *J. Warga*, Optimal Control of Differential and Functional Equations. Academic Press, New York, 1972.
35. *T. Weber*, An infinite-horizon maximum principle with bounds on the adjoint variable, *Journal of Economic Dynamics & Control*, 30, 229-241, 2006.
36. *J. J. Ye*, Nonsmooth Maximum Principle for infinite-horizon problems, *J. Optim. Theory Appl.*, 76, 485-500, 1993.